Turnstile Figures of Opposition

Jean-Yves Beziau

Abstract We present many figures of opposition (triangles and hexagons) for simple and double turnstiles. We start with one-sided turnstiles, corresponding to sets of tautologies, then we go to double-sided turnstiles corresponding to consequence relations. In both cases we consider proof-theoretic (with the simple turnstile) and model-theoretic (with the double turnstile) figures. By so doing we discuss various central aspects of notations and conceptualizations of modern logic.

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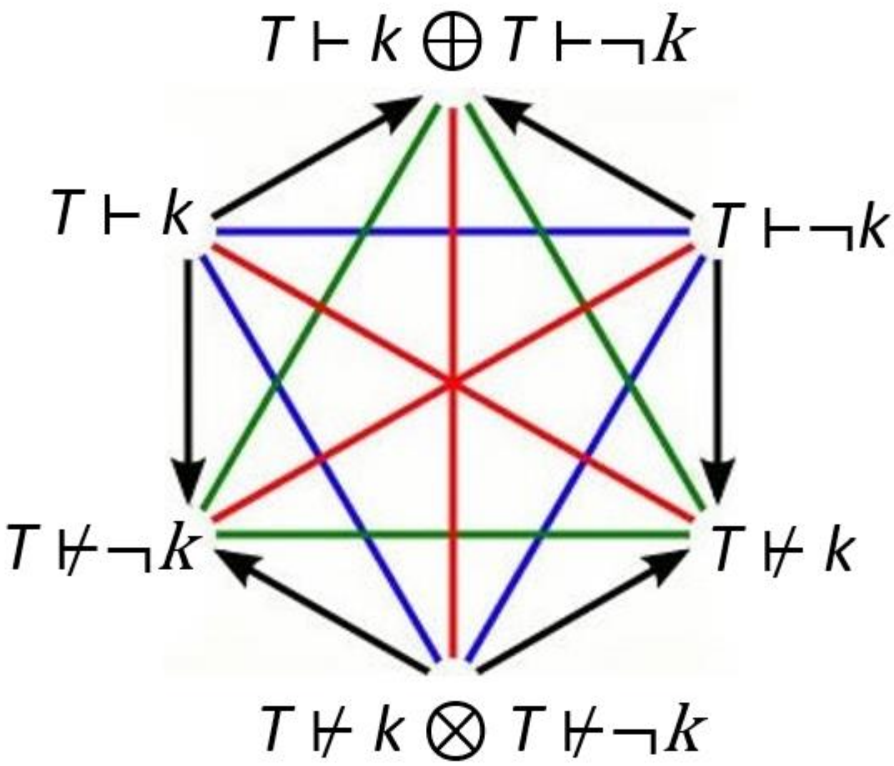
Keywords Square of Opposition, Turnstile, Tautology, Truth, Proof, Consequence, Model.



1 The Hexagon of Opposition and the Turnstile

The hexagon of opposition was introduced by Robert Blanché.[1] It is an improvement or/and reconstruction of the famous square of opposition. Here is a picture of it:

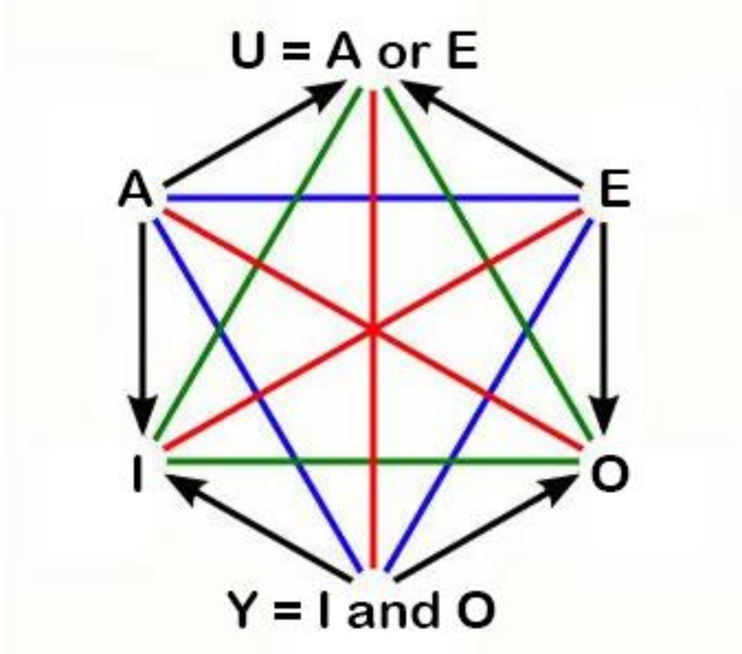


FIGURE 1 – HEXAGON OF OPPOSITION

We have the same four relations as in the square: the black arrow is the relation of subalternation, in red we have the relation of contradiction, in blue the relation of contrariety and in green the relation of subcontrariety. We recall the basic definitions: two propositions are said to be *contradictory* iff they cannot be true and cannot be false together, *contrary* iff they cannot be true but can be false together, *subcontrary* iff they cannot be false but can be true together. *Subalternation* is an implication.

In the above hexagon we can find the traditional square of opposition with corners A, E, I, O. Blanché introduced two additional corners that he named U and Y and which are defined as indicated. In the hexagon we can see two additional squares of opposition: Y, A, U, O and E, Y, I, A, as well as a contrariety triangle in blue and a subcontrariety triangle in green.

The hexagon of opposition has been applied to many topics ranging from deontic notions to the theory of colors, through music, economy and quantum physics (cf. [25], [27] and other papers in the many volumes of collected papers ([13], [14], [17], [18]) and special issues ([12], [15], [16], [19]) which have been

[1] His main book on the topic is [20], but his first works were published in the 1950s, and at this time other people had similar ideas (for details, see [3]).

published since the revival of the square (cf. [1], [4]) and the *1st World Congress on the Square of Opposition* in Montreux in 2007. It can also be applied to the theory of opposition itself (see [8]).

Here we will apply it to logical notions. This paper is a follow up of my paper "The metalogical hexagon of opposition" [6]. It is also related with the talk "Beyond Truth and Proof" I gave in Tübingen at the workshop *Consequence and Paradox Between Truth and Proof (March 2-3, 2017)* and the tutorial I gave at UNILOG'2018 (*World Congress and School on Universal Logic*) in Vichy in June 2018: "The Adventures of the Turnstile".

The present paper connects two aspects of symbolism: diagrammatic symbolism and the use of non-alphabetical signs. In logic, among the first category, the square is the most famous representative followed by Venn diagrams. Among the second category we have in particular the connectives: "$\vee$, $\neg$, $\wedge$, $\rightarrow$" and the quantifiers: "$\forall$, $\exists$". But probably the most famous one is "$\vdash$".

This symbol was introduced by Frege (1879, cf. [22]) with a specific meaning that we will not discuss here (see e.g. [30]). It is nowadays used with another meaning which is not always clear. The aim of this paper is to clarify the contemporary meaning(s) of "$\vdash$", using the theory of opposition, in particular triangles of contrariety and hexagons of opposition.

Doing that we will deal with the sister symbol "$\vDash$", which is called the *double turnstile*, by contrast to "$\vdash$", called *simple turnstile*. "$\vdash$" is also called *Frege's stroke*, but we will not use here this terminology, because on the one hand we are not dealing with the original meaning given to it by Frege and on the other hand the turnstile terminology is nice because it allows to use the same word "turnstile" to qualify two different connected notions. It would make no sense to talk about Frege's *simple stroke* and *Frege's double stroke*, since Frege did not introduce "$\vDash$" (this symbol was introduced in the 1950s).

2 Tautological figures of opposition

2.1 Two pretty different contrariety triangles

There is the dichotomy between truth and falsity, that we find in particular in classical propositional logic. We can go beyond this dichotomy by adding a third value, or more values. This in particular what Łukasiewicz [26] did, inspired by Aristotle. If we have three-values, we have then the following triangle of contrariety:

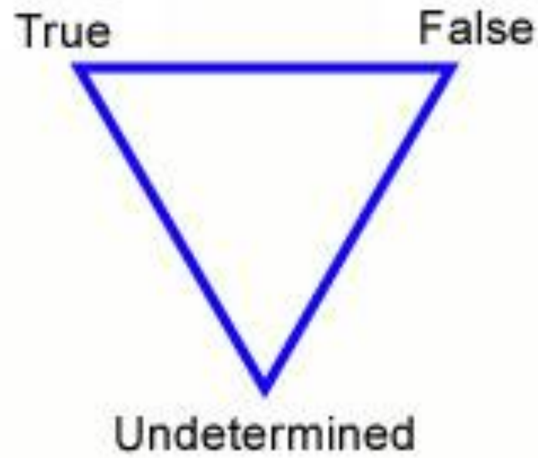


FIGURE 2 – THREE-VALUED TRIANGLE OF CONTRARIETY

On the other hand, there is a distinction which is at another level. This is the distinction between truth and logical truth, promoted in particular by Wittgenstein [35] putting forward the notion of tautology. This leads to a subtler triangle of contrariety:

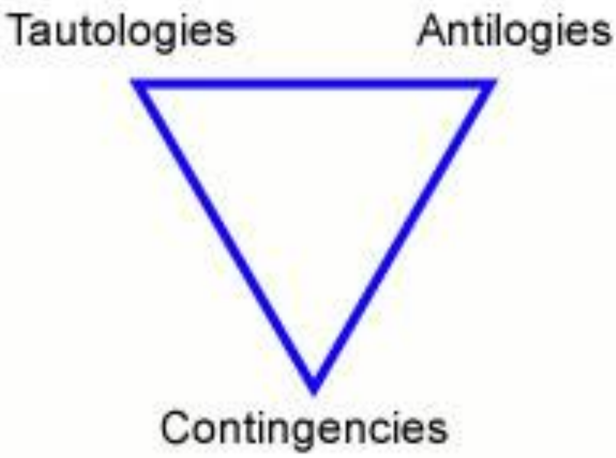


FIGURE 3 – TAUTOLOGICAL TRIANGLE OF CONTRARIETY

Wittgenstein didn't use the words "antilogies" and "contingencies". This terminological choice is explained in [6], [8] and [9]. But although we are using a different terminology, we are presenting here the same trichotomy as in the *Tractatus*: a tautology is always true, an antilogy is always false, a contingency can be true and can be false.

2.2 Symbolic representation of the tautological triangle

Wittgenstein presented the trichotomy among propositions using a framework which is nearly identical to the one used nowadays for the semantics of classical propositional logic based on valuations (that he calls "truth-possibilities" (*Tractatus* 4.3). We can present it in the following table:

TERMINOLOGY	MATHEMATICAL DEFINITION
Tautology	$\forall v\ \ v(p)=1$
Antilogy	$\forall v\ \ v(p)=0$
Contingency	$\exists v\ \ v(p)=1$ and $\exists v\ \ v(p)=0$

TABLE 1

We have called what is on the right column "mathematical definition" to emphasize that it is not just symbolism. In modern logic, mathematical tools, objects, concepts are used. Here for example the numbers 0 and 1 (not only the notations "0" and "1") are used, as well as the notions of function and equality.[2]

In modern mathematical logic the expression "$\forall v\ \ v(p)=1$" is also written "$\vDash p$". The latter can be seen as an abbreviation of the former, it is indeed shorter. "$\forall$" can also be seen as an abbreviation of "All". This is the first letter of this word put upside down (notation introduced by Gentzen, following the same idea as for the sign of the existential quantifier "$\exists$" introduced by Peano). But mathematical writing is not only a question of abbreviation. There is the idea to use signs which are not completely arbitrary, that have a serious symbolic aspect in the true sense of the word (see [31] and [7]). The sign "$\vdash$" was introduced by Frege with a real symbolic dimension expressing an important distinction through perpendicularity. "$\vDash$" is a symbol directly inspired by "$\vdash$". The similar graphic design of the two signs expresses the connection between their meanings, and the difference of meaning is expressed by doubling the horizontal line. This is nicely reflected in natural language by the expressions *simple turnstile* and *double turnstile.* Natural language is useful in particular when talking.

After having established a correspondence between "$\forall v\ \ v(p)=1$" and "$\vDash p$", how can we go further on, rewriting the other mathematical definitions

[2] Wittgenstein uses "F" and "W", not "0" and "1". In general his framework is not explicitly mathematical, although he uses the notion of function, following Frege and Russell. About 0 and 1 as truth-values, the notion of truth-function, etc., see [10] and [5].

using the double turnstile? There is no "direct" way to do that. The best we can do with "$\forall v \ \ v(p)=0$" is to write it as "$\vDash\neg p$" considering the definition of classical negation according to which $v(p)=0$ iff $v(\neg p)=1$.

It is even less straightforward to express contingency with the double turnstile. We have to use the symbol "$\nvDash$", which uses a negation at the metalevel. According to that, "$\nvDash p$" means $\exists v \ \ v(p)=0$ (we are not putting quotes here, because we are not talking about this symbolic formula but about its meaning: p is false according to one valuation). "$\nvDash p$" is the syntactic (metalevel syntax) negation of "$\vDash p$", which itself means $\forall v \ \ v(p)= 1$. Here we have to be very careful because there is a mix between logic and metalogic. $\exists v \ \ v(p)=0$ is the negation of $\forall v \ \ v(p)=1$ at the metalogical level (again we don't use quotes here because we are not talking about "$\exists v \ \ v(p)= 0$" and "$\forall v \ \ v(p)= 1$", but about their meanings).

The ambiguity is that the symbols "$\exists$" and "$\forall$" are generally used as symbols for quantifiers in first-order logic, at a logical level, not at a metalogical level. Here we are using them at a metalogical level. One may think that the meta-theory of propositional logic (classical or not) can be carried out in first-order logic. This is true up to a certain point. But it is not necessarily obvious, details have to be checked, and someone may defend another point of view. Here we stay neutral. If we use the symbols "$\exists$" and "$\forall$" it is rather a question of abbreviation. There are no other symbols standardly used for that, like in the case of implication where we can make the distinction between implication and meta-implication respectively using the symbols "$\longrightarrow$" and "$\Longrightarrow$"[3]. Likewise, in the case of conjunction and meta-conjunction, we can make the distinction using the symbols "$\wedge$" and "&".

Using the latter symbol we can rewrite the mathematical definition of contingency as "$\exists v \ \ v(p)=1$ & $\exists v \ \ v(p)=0$", which in turn we can express using the double turnstile: "$\nvDash p$ & $\nvDash \neg p$". At the end, using also negation at the metalevel for the second part of the mathematical definition of contingency we have the following table:

TERMINOLOGY	MATHEMATICAL DEFINITION	DOUBLE TURNSTILE
Tautology	$\forall v \ \ v(p)=1$	$\vDash p$
Antilogy	$\forall v \ \ v(p)=0$	$\vDash\neg p$
Contingency	$\exists v \ \ v(p)=1$ & $\exists v \ \ v(p)=0$	$\nvDash p$ & $\nvDash \neg p$

TABLE 2

[3] The difference between the two levels is expressed here by doubling the horizontal line. For the turnstile, the doubling of the horizontal line is not used in this sense.

Base on the right column, we can represent the triangle of contrariety of Fig. 3 in the following manner:

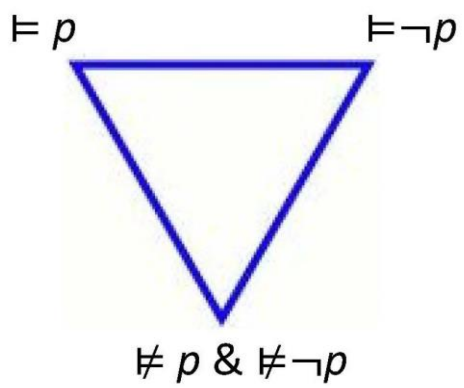


FIGURE 4 – DOUBLE TURNSTILE TAUTOLOGICAL TRIANGLE OF CONTRARIETY

Nowadays there is a clear distinction between "$\vDash$" and "$\vdash$", the latter being used in proof theory (also called syntax) by contrast to the former used in model theory (also called semantics). For classical propositional logic the bridge was established by Emil Post in a paper published in 1921 [29], the same year of the publication of Wittgenstein's *Tractatus*,[4] and in 1930 by Kurt Gödel for first-order logic [23]. We have therefore the following table:

TERMINOLOGY	DOUBLE TURNSTILE	SIMPLE TURNSTILE
Tautology	$\vDash p$	$\vdash p$
Antilogy	$\vDash\neg p$	$\vdash\neg p$
Contingency	$\nvDash p$ & $\nvDash \neg p$	$\nvdash p$ & $\nvdash\neg p$

TABLE 3

Accordingly, the triangle of Fig 4. is equivalent to the following one:

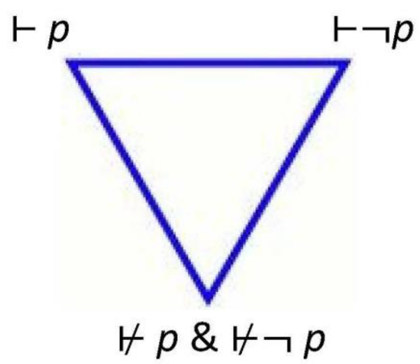


FIGURE 5 – SIMPLE TURNSTILE TAUTOLOGICAL TRIANGLE OF CONTRARIETY

[4] Post was using only "$\vdash$". As we said, "$\vDash$" was introduced in the 1950s. Wittgenstein was using none of these symbols, he rejected Frege's stroke (cf. *Tractatus* 4.442).

Using the simple and double turnstiles we are able to explicitly make the distinction between proof theory and model theory, truth and proof, syntax and semantics, a distinction which is not clearly made at the level of natural language. For example, the word “tautology” is not clearly attached to one of the side of the dichotomy. One may say that is not important because of the completeness theorem. But in fact the distinction is important. If we don’t make the distinction, the completeness theorem has no meaning. Moreover, if we have a general perspective, being interested not only in classical propositional logic, but in many other systems of logic, there are some cases where the completeness theorem does not hold.

The two triangles of Fig 4 and Fig 5 clearly show the general structure of a triangle of contrariety. The bottom corner is the conjunction of the (metalogical) negations of the two top corners. The position of the decorations of the corners is mostly irrelevant, contrary to the spirit of the traditional theory of opposition with the labels “A”, “E”, “I”, “O” for the corners of the square, which are moreover connected to a special version of the square, i.e. the original square of categorical propositions.

We have put on the top left corner the notion of tautology, and the corresponding notations: “$\vDash p$” and “$\vdash p$”. The reason to do so is because it is the most famous notion. This is also the reason why we have called these triangles “tautological triangles”. Other words are used for the notion of tautology, for example *logical truth*, but *tautology* is more striking. It is usual to call a figure of opposition (a triangle, a square, a hexagon) by the name of one of its corners, e.g. the analogical hexagon [11]. Another option is to use the name of the family of notions involved in the figure, i.e. the deontic hexagon. Here we could have used the expression “metalogical triangle” as we did in [6]. The reason not to do that here is that we want to make explicit the distinction between two metalogical figures, the one corresponding to logics as sets of tautologies and the one corresponding to logics as consequence relations, both are metalogical.

Another distinction is between propositional logic and first-order logic. Our two triangles of Fig.4 and Fig.5 can be seen from both perspectives. But on the one hand in our tables we have given only the mathematical definitions corresponding to propositional logic, and on the other hand the use of the letter “*p*” is generally attached to propositional logic by contrast to first-order logic.

What we want to do here is to make a uniform presentation for all variations of classical logic: propositional, first-order, second-order, etc. We are also aiming at a very general framework not limited to classical logic. The only typical feature of classical logic we are using is classical negation.

Therefore, our framework applies to extensions of classical logics such as modal logics or non-classical logics with a classical negation such as some paraconsistent logics.

We can make the following new version of TABLE 2:

TERMINOLOGY	MATHEMATICAL DEFINITION	DOUBLE TURNSTILE
Tautology	$\forall\mathcal{M}\ \mathcal{M} \vDash k$	$\vDash k$
Antilogy	$\forall\mathcal{M}\ \mathcal{M} \nvDash k$	$\vDash \neg k$
Contingency	$\exists\mathcal{M}\ \mathcal{M} \vDash\ k\ \&\ \exists\mathcal{M}\mathcal{M} \nvDash k$	$\nvDash k\ \&\ \nvDash \neg k$

TABLE 4

When we write "$\mathcal{M} \vDash k$" we are using the double turnstile in a different way as on the right column. It is a bit ambiguous, but is based on a link between the two: The meaning of "$\vDash k$" is defined by $\forall\mathcal{M}\ \mathcal{M} \vDash k$. Generally "$\mathcal{M} \vDash k$" is used only in first-order logic but Chang and Keisler in their famous book *Model Theory* [21] also use this notation for propositional logic. However they don't use the letters "$\mathcal{M}$" and "k", they use another notation.

On the left side we are using the letter "$\mathcal{M}$" using a graphism different from the one of the letter "k" to emphasize that these are different kinds of entities. We use the 13th letter of the alphabet because it is the initial letter of the word "model". This word can be used in any context because it does not specify the internal nature of the thing, whether it is a bivaluation, a first-order structure, a possible world, but only its function. It is a bit ambiguous because if "$\mathcal{M} \vDash k$" can be read without problem as "$\mathcal{M}$ is a model of k", on the other hand "$\mathcal{M} \nvDash k$" is read as "$\mathcal{M}$ is not a model of k", which is a bit paradoxical, because we have a model which is not a model ! But this makes sense if we consider that $\mathcal{M}$ is a model of other formulas, here for example of $\neg k$.

Instead of writing "$\mathcal{M} \vDash k$" and "$\mathcal{M} \nvDash k$" we could have respectively written "v($\mathcal{M}$; k)=1" and "v ($\mathcal{M}$; k)=0" but it would have been a bit cumbersome. Anyway "$\mathcal{M} \vDash k$" is usually read as "k is true in $\mathcal{M}$" and "$\mathcal{M} \nvDash k$" as "k is false in $\mathcal{M}$". What is important to stress is that a principle of bivalence is used at the metalogical level whether we are dealing with a non-classical logic or/and a first-order logic.

Why using the letter "k"? We want to avoid to use the letter "p" which is too much connected to propositional logic, so we chose the 11th letter of the alphabet which is quite neutral. We could have chosen the letter "f", considering that it is the first letter of the word "formula". This word is quite neutral and is used to talk either of formulas of propositional logic or first-order logic. But this word has an ambiguous meaning: it is also used for any symbolic expression (not necessarily connected to logic). It is important to emphasize the nature of the object we are dealing with and to which the three categories tautology, antilogy contingency apply. These are propositions, whether specified as formulas of a propositional language or another formal language. So we will keep using the word "proposition", but we prefer to use "*k*" than "*p*" to avoid the reader to immediately think that we are dealing only with propositional logic.

We have then the two following diagrams:

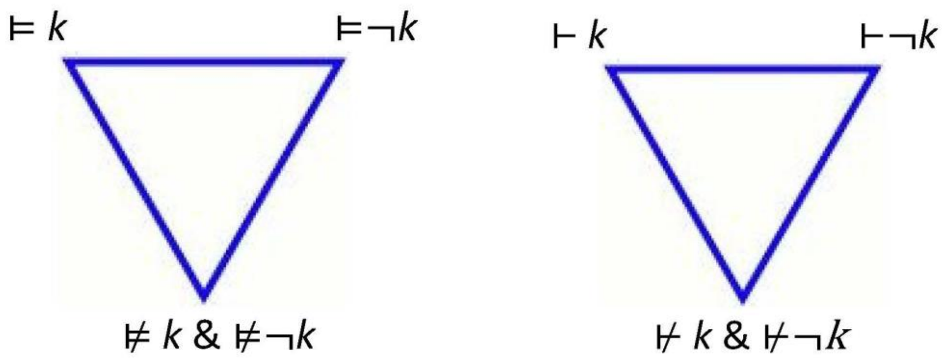


FIGURE 6 – TURNSTILES TAUTOLOGICAL TRIANGLES OF CONTRARIETY

Based on TABLE 4, the left diagram can be designed as follows without using negation at the logical level:

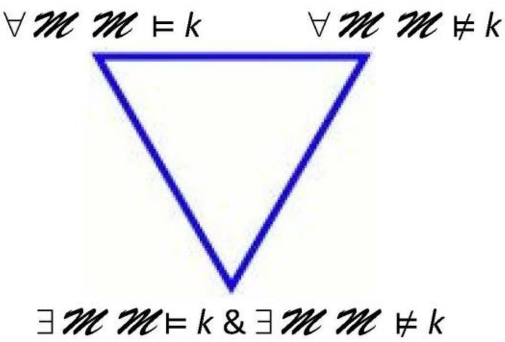


FIGURE 7 – MODEL-THEORETIC TRIANGLE OF CONTRARIETY

It is interesting because we have then a diagram not limited to logics with a classical negation, such as positive logic. This triangle could be called a "turnstile triangle" since the (double) turnstile is used, but not with the same meaning as in Fig. 4.

2.3 Turnstile tautological hexagons

Let's now apply the structure of the hexagon. We have then the following diagrams:

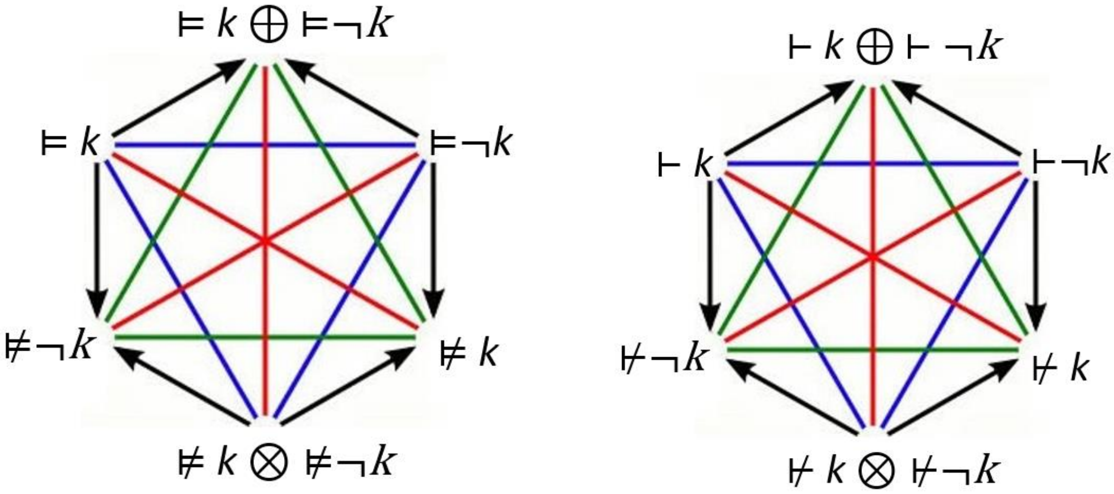


FIGURE 8 – TURNSTILES TAUTOLOGICAL HEXAGONS

These hexagons have been generated using the logical structure of this figure of opposition. The three corners of the green triangles of subcontrariety are the (metalogical) negations of the three corners of the blue triangles of contrariety. We are using the symbol "$\oplus$" to denote metalogical disjunction. And we have replaced "&" by "$\otimes$" for metalogical conjunction. This is not only purely esthetical. A good notation has to be designed considering the general context, in relation with other notations. For example the symbol for the empty set "$\varnothing$" (introduced by André Weil) is a good notation considering its link with the symbol for the number zero "0". It is good to have a connection between the symbols for conjunction and disjunction. At the logical level we have "$\wedge$, $\vee$" and that's nice. At the metalogical level we also chose here two symbols having a connection (and multiplication and addition are traditionally connected with conjunction and disjunction).

"$\nvDash$ *k*" can literally be interpreted as: *k* is not a tautology, which means nothing else than: *k* is an antilogy or *k* is a contingency, as clearly depicted by the structure of the hexagon. There is not positive terminology for this situation, maybe it could be good to create one. The same happens with the two other cases: the contradictory opposite of antilogy and the contradictory opposite of contingency.

3. Hexagons of opposition for consequence relations

At some point a logic started to be considered as a consequence relation rather than a set of tautologies. The origin of this framework can be traced back to Tarski when in Poland. He put forward on the one hand the notion of consequence operator [33], on the other hand the notion of logical consequence [34]. In both cases we have a binary setting: a formula is consequence of a set of formulas. These two notions studied by Tarski are not defined in the same way and he didn't use the same terminology for them.

Tarski at this time was using neither "⊢" nor "⊨". Nowadays it is common to use these symbols as binary relations in the following way:

SYMBOLISM	READING	MEANING
$T \vdash k$	k is a proof-theoretic consequence of T	There is a proof of k from T
$T \vDash k$	k is a model-theoretic consequence of T	All models of T are models of k

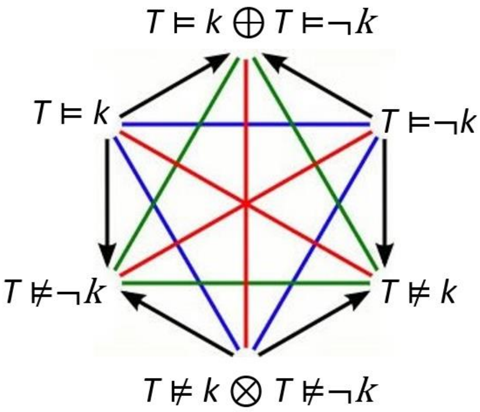
TABLE 5 – PROOF-THEORETIC AND MODEL-THEORETIC CONSEQUENCE RELATIONS

In continuity with what we have said in the previous section, what is on the right of the simple or double turnstile, we will call it a *proposition* and denoted it by "k". On the left side we have what is generally called a *theory*,[5] a set of formulas, or to use our present language, a set of propositions. We use a capital letter for a theory to emphasize the difference between the size: multiplicity vs. oneness. Multiplicity on the right of the turnstile has also been considered (cf. [32]) but we will not deal here with this issue.

In the case of both turnstiles, the tautological framework can be seen as a particular limit case of the consequential framework, the case where the theory is the empty set: $\varnothing \vdash k$ and $\varnothing \vDash k$.

Symbolically we have then the two following consequential turnstile hexagons of which the two hexagons of Fig. 8 are limit cases:

[5] In Poland during the thirties the word "theory" was used in a different way: for what is nowadays called a "closed theory", a theory such that any formula which is a consequence of the theory is in the theory.



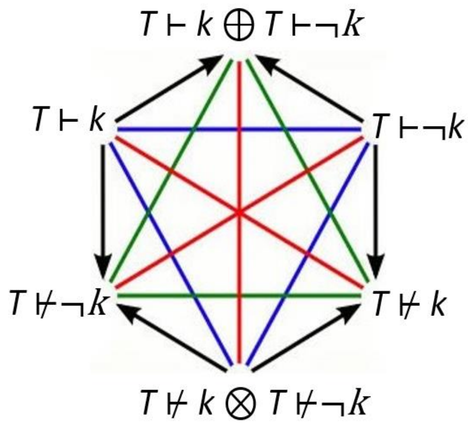


FIGURE 9 – TURNSTILE CONSEQUENTIAL HEXAGONS

These hexagons perfectly depict the 6 possibilities we have for a relation of a proposition relatively to a theory, either from a model-theoretic point of view (on the left) or from a proof-theoretic point of view (on the right). The completeness theorem can be interpreted as the matching of these two hexagons.

These 6 positions do not always exist. For example, if we have a *complete theory*, the bottom position does not exist. The definition of a complete theory is given by the top position. A famous case of incomplete theory is Peano Arithmetic, *PA*. Gödel [24] has shown that there is a proposition *g*, inspired by the liar paradox, such that $PA \nvdash g$ and $PA \nvdash \neg g$. Sometimes such a proposition is called an *undecidable* proposition, but a better terminology is *independent.*[6]

Let's see what kind of names we can give to the other positions. We can design the following hexagon:

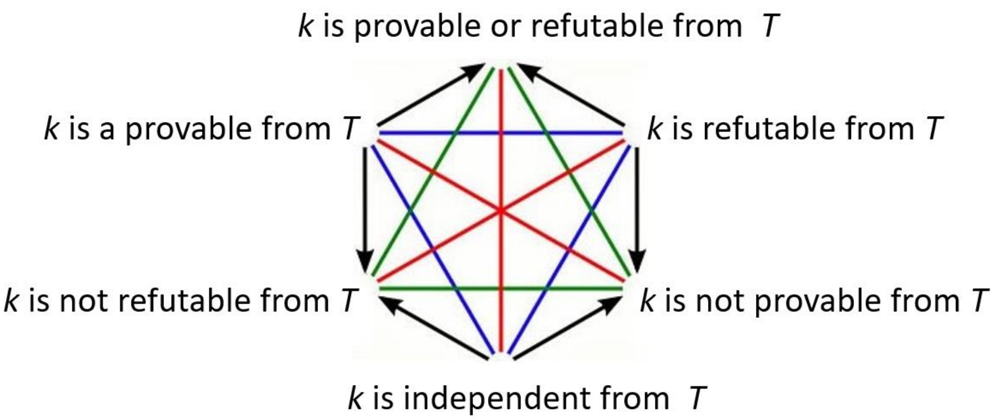


FIGURE 10 – TERMINOLOGICAL PROOF-THEORETIC HEXAGON

[6] A theory can be incomplete and decidable, a famous case is the empty theory of classical propositional logic, an atomic formula is independent from ∅, but ∅ is decidable.

We clearly have some positive terms for the three corners of the contrariety triangle. It is no clear that we can find some non-ambiguous terminology for the three other corners. But note that in this figure we have avoided to use negation at the logical level, so it can apply to any logical system.

Now let's turn to the model-theoretic hexagon. We have the following table:

	MATHEMATICAL DEFINITION	DOUBLE TURNSTILE
A	$\forall \mathscr{M}\ \mathscr{M} \vDash T \Longrightarrow \mathscr{M} \vDash k$	$T \vDash k$
E	$\forall \mathscr{M}\ \mathscr{M} \vDash T \Longrightarrow \mathscr{M} \nvDash k$	$T \vDash \neg k$
Y	$(\exists \mathscr{M}\ \mathscr{M} \vDash T \otimes \mathscr{M} \nvDash k) \otimes (\exists \mathscr{M}\ \mathscr{M} \vDash T \otimes \mathscr{M} \vDash k)$	$T \nvDash k \otimes T \nvDash \neg k$
I	$\exists \mathscr{M}\ \mathscr{M} \vDash T \otimes \mathscr{M} \vDash k$	$T \nvDash \neg k$
O	$\exists \mathscr{M}\ \mathscr{M} \vDash T \otimes \mathscr{M} \nvDash k$	$T \nvDash k$
U	$\forall \mathscr{M}\ \mathscr{M} \vDash T \Longrightarrow \mathscr{M} \vDash k \oplus \forall \mathscr{M}\ \mathscr{M} \vDash T \Longrightarrow \mathscr{M} \nvDash k$	$T \vDash k \oplus T \vDash \neg k$

TABLE 6 –MODEL-THEORETIC CONSEQUENTIAL HEXAGON

This allows us to have a consequential hexagon with the use of negation only at the metalevel similarly to the triangle presented in Fig. 7:

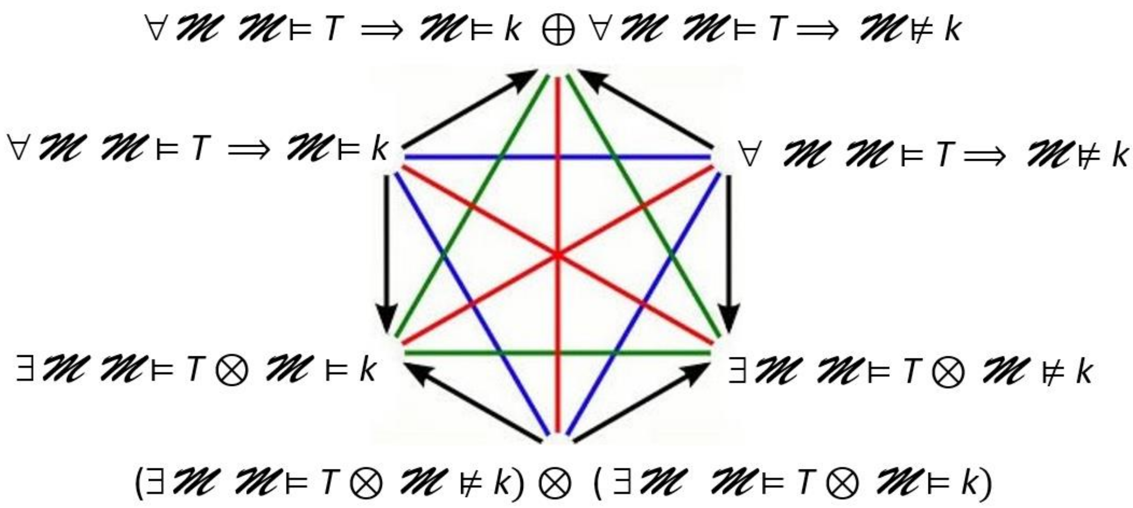


FIGURE 11 – MODEL-THEORETIC HEXAGON

If we want to use a truth terminology, we can interpret "$T \vDash k$" as "k is true in T" or "k is true according to T" for example "2+2=4" is true according to Peano Arithmetic. And we can interpret "$T \vDash \neg k$" as "k is false in T" or "k is false according to T". For example, $2+2\neq 4$ is false according to Peano Arithmetic. Up to now, no problems. From this point of view the Y corner of the hexagon can be interpreted as neither true nor false in T (or according to T), but there is no straightforward terminology to summarize this in one word. And we can interpret the other corners of the hexagon in a pure negative way. We then have the following diagram:

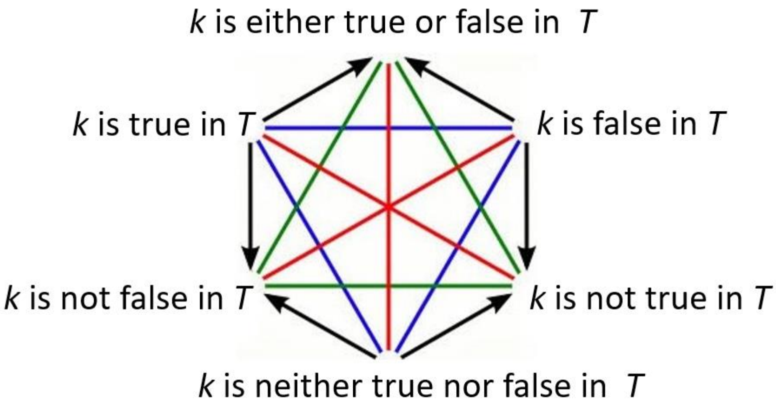


FIGURE 12 – TERMINOLOGICAL MODEL-THEORETIC HEXAGON -TRUTH VERSION

A very important point is that "*k* is true in *T*" is not equivalent here to "*k* is not false in *T*". Symbolically: $T \vDash k$ is not equivalent to $T \nvDash \neg k$. At the level of symbolism it is interesting because we see that we have a logical negation and a meta-logical negation, and the two together do not lead to affirmation.

We may want to eliminate truth, then we can design the following diagram:

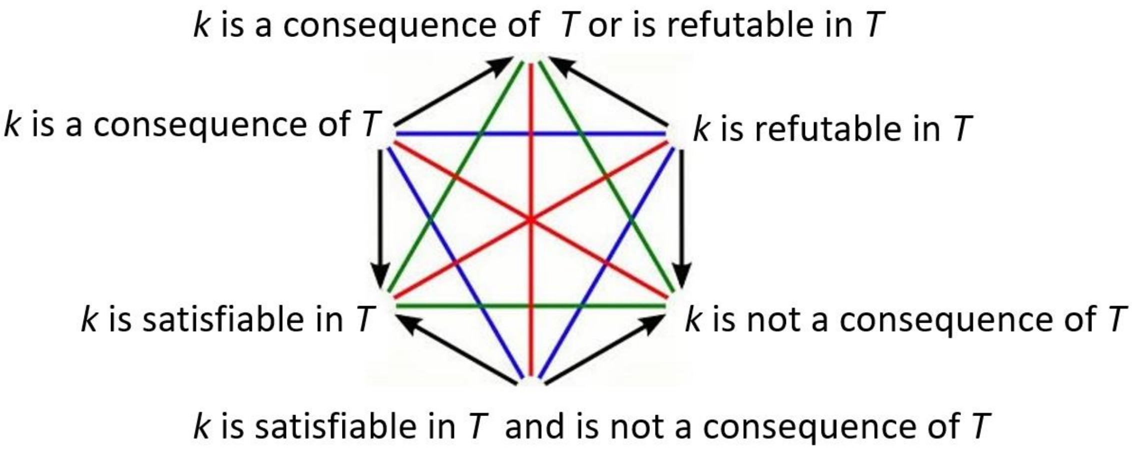


FIGURE 13 – TERMINOLOGICAL MODEL-THEORETIC HEXAGON - NO TRUTH VERSION

The word "satisfiable" is clearly from model-theory, but generally it is not used in this way: we say that a formula is satisfiable in a model, not in a theory. The terminology "satisfiable" is quite natural for the I-corner when *T* is empty, i.e. in the case of the tautological model-theoretic hexagon. Then we say that a formula is satisfiable *tout court*.

On the E-corner we have put “refutable” which is rather from proof theory. We have used “in” rather than “from” to have a similar expression as with satisfiability and different from the proof-theoretic hexagon of Fig. 10.

From the point view of model theory, it would make more sense to put refutable in the O-corner, where we have put “is not a consequence”. Again, this is natural in the empty case, when we say refutable *tout court*. This is the reason why in our previous paper [6] we put refutable in the O-corner forming a nice subcontrariety pair with satisfiable rather than a contradictory pair as in Fig. 13. The problem we are facing here is that in proof theory it makes also sense to put it in the E-corner.

To finish, let us present a new terminological decoration of the six corners of the consequential hexagon:

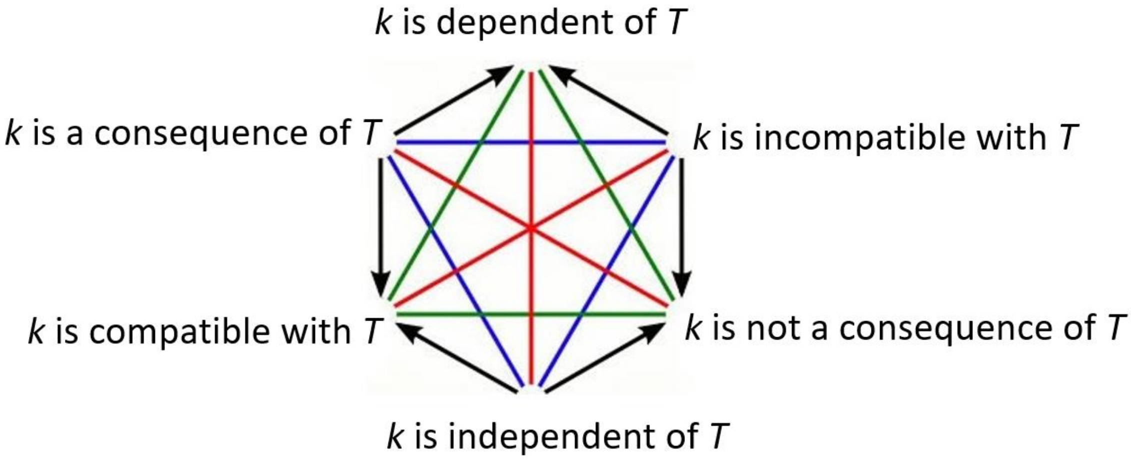


FIGURE 14 – TERMINOLOGICAL CONSEQUENTIAL HEXAGON

As in Fig. 10, 11 12 and 13, we have avoided to use negation at the logical level, so this figure applies to any logical system. Moreover, the advantage of the terminology of this diagram is that it can be used both for proof theory and for model theory. The terminologies “(in)compatible” and “(in)dependent” are not usually univocally tight to one of these fields. This advantage turns of course into a defect if we want to emphasize one of the two specific fields.

Considering the turnstile symbolism, the two fields are clearly distinguished by the simple turnstile “⊢” and the double turnstile “⊨”. There is not a symbol which is unambiguously used to deal with an abstract situation which is beyond proof and truth, although in recent years the tendency has been to use the simple turnstile for such a situation.

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Jean-Yves Beziau
University of Brazil, Rio de Janeiro
Brazilian Research Council
Brazilian Academy of Philosophy
e-mail: jyb@ufrj.br