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IDENTITY, STRUCTURE AND LOGIC

0. Introduction

The concept of identity is a central idea of thought. We will see how modern mathematics can give a precise analysis of this concept and how logic can deal with this analysis.

We will define three kinds of identity: the Bourbaki identity, the logical identity and the diagonal identity (in short B-, l-, d-identity respectively) and study the connections between them. A whole picture of these relations is given at the end of the paper.

In a given structure, two objects are B-identical if they have the same position in the structure. The question is: under which conditions these two objects are the same? The answer is: in case the structure is rigid, i.e. has no non-injective endomorphism.

Two objects are l-identical i they obey the same formulas. If we have a truth-functional logic, this notion coincides with the B-identity. In fact this coincidence takes place for every Fregean logic.

What happen if we put a non-Fregean logic, e.g. the paraconsistant logic $C1^*$, on the structure? There is a strange phenomenon: two isomorphic structures are not necessarily elementary equivalent. We find a solution to this problem in adapting our result on rigidity: we give a new definition of morphism, logical morphism which preserves not only atomic statements but the whole logical hierarchy of compound statements.

In fact the notion of structure is also generalized: a structure is inseparable from the set of all its true compound statements, this is what we call a logical structure.

1. Ontology of the structure

According to the ontology of modern mathematics, especially developed by Nicolas Bourbaki (see [1]), *to be is to be an object of a structure*. It means that the *nature* of an object is entirely determined by the structure it is merged into.

An object is *determined* by its *situation* in the structure: by the *relations* to the other members of the structure.

Let us take the example of a structure of type $\mathcal{S} = \,< S; R >$ where S is a set and R is a binary relation on S. (All we will say, for the sake of clarity and simplicity, will be based on this example; it is easy to see that this can be generalized to any kind of structure.)

We define the *identity* of an object a of $\mathcal{S}$, ID[a], in the following way. We first define the *left identity* of a: IDL[a] = $\{x \in S : \,< x, a > \,\in R\}$ and the *right identity* of a: IDR[a] = $\{x \in S : \,< a, x > \,\in R\}$, and then we put: ID[$a$] = <IDL[$a$];IDR[$a$]>.

Two objects of the structure $\mathcal{S}$ are called *B-identical* i they have the same identity.

2. Rigidity

It is clear that in general two objects can be B-identical without being one and the same object.

First we will say exactly what ≪one and the same object≫ means, and second, in which conditions two objects are B-identical i they are one and the same object.

By the d-identity we mean the diagonal of $S \times S$: two objects are said to be *d-identical* i they belong to the diagonal of $S \times S$ (i.e. the pair of these two objects is in the diagonal).

Let us call by a *proper endomorphism* an endomorphism which is not injective. We will say that a structure is *rigid* i it has no proper endomorphism.

THEOREM 1. *The d-identity and the B-identity coincide in a given structure i it is rigid.*

PROOF.

($\Rightarrow$): Suppose that the Bourbaki and diagonal identities coincide in a structure $\mathcal{S}$ and that there exists a proper endomorphism ε of $\mathcal{S}$. Then for some a, b

(1) $\varepsilon a = \varepsilon b$, and $< a, b >$ is not in the diagonal of S, so

(2) a, b are B-identical.

From (2) it follows that for some c, for example $< a, c > \in R$ and $< b, c > \notin R$. Then $< \varepsilon a, \varepsilon c > \in R$ and $< \varepsilon b, \varepsilon c > \notin R$ which is impossible due to (1).

($\Leftarrow$): Suppose that di erent a, b are B-identical, and define the following function ε on S: $\varepsilon a = \varepsilon b = a, \varepsilon c = c$ for $c \in S - \{a, b\}$. Clearly ε is not 1–1. From the assumption it follows that

(3) $< a, x > \in R$ i $< b, x > \in R$

(4) $< x, b > \in R$ i $< x, a > \in R$, any $x \in S$.

Then due to the definition of ε, the condition $< x, y > \in R$ i $< \varepsilon x, \varepsilon y > \in R$ holds for every $x, y \in S - \{b\}$. Furthermore for $x \neq b$: $< x, b > \in R$ i $< x, a > \in R$ i $< \varepsilon x, \varepsilon b > \in R$ by (4) and the definition of ε.

Similarly: $< b, x > \in R$ i $< \varepsilon b, \varepsilon x > \in R$ holds true. In the last case, $< b, b > \in R$ i $< a, b > \in R$ i $< a, a > \in R$ i $< \varepsilon b, \varepsilon b > \in R$ by (3), (4) a and the definition of ε. $\square$

3. The logical step

When we assert that a is in relation R to b we have the statement aRb about a structure and we give to it a truth value, the truth: $tv(aRb) = 1$. Now the problem is: what does it mean when we assert: it is not the case that a is in relation R to b. Does it mean $tv(aRb) = 0$ or $tv(\neg(aRb)) = 1$? In classical logic it is exactly the same because $tv(aRb) = 0$ i $tv(\neg(aRb)) = 1$. But in a paraconsistent logic we can have $tv(aRb) = 1$ and $tv(\neg(aRb)) = 1$. Thus if we interpret that a is not in relation R to b as $tv(aRb) = 0$ it is not the same as in case we interpret it as $tv(\neg(aRb)) = 1$ because in this case it could be that $tv(aRb) = 1$.

We can define the B-identity and the notion of endomorphism in such a way that Theorem 1 will be valid for any logic. We say that two objects a and b are *B-identical* i for every $x : tv(aRx) = tv(bRx)$ and $tv(xRa) = tv(xRb)$. And a function ε from S to S is an *endomorphism* i $tv(aRb) = tv(\varepsilon aR\varepsilon b)$.

3.1. Logical structure

If we want to developed a general study of structures from the point of view of any logic as suggested by N. C. A. da Costa, an idea which is correlated with his idea of polyvalent platonism (see [2], p. 192), we must change the definition of structure.

The usual presentation of a structure is atomistic in the following way: when, for example, we speak about the structure $\mathcal{S} = <S, R>$, R can be viewed in fact as a subset of atomic statements, the true atomic statements.

Now by a *logical structure*, we mean a triple $<S; L; tv>$, where S is a set, L is a set of statements about the elements of S, and tv is a function from L to $\{0,1\}$. L is supposed to be a set of atomic statements relative to a binary relation R or S as well as compound statements constructed from the atomic ones, like negations or quantifications, e.g. $\forall x(xRa)$ is a statement about a. It does not contain the atomic statements of the form: $a = b$. The same concept of d-identity, previously considered, is preserved for a logical structure.

Given a logical structure, we say that two objects a and b are *l-identical* i $tv(\phi(a)) = tv(\phi(b))$, for every statement ϕ.

It is easy to see that, in general, the l-identity does not necessarily coincide with the Bourbaki one. Let us take again the example of paraconsistent logic. We may have: $tv(\neg(aRc)) = 1$ and $tv(\neg(bRc)) = 0$, thus a and b are not l-identical, but they can be B-identical since we can have $tv(aRc) = 1$ and $tv(bRc) = 1$.

3.2. Fregean structure

Now we will show a condition under which the l-identity can be reduced to the Bourbaki one.

We say that a logical structure is a *Fregean structure* i $tv(\phi) = tv(\psi)$ implies $tv(\chi) = tv(\chi[\phi/\psi])$, where ψ is a substatement of the statement χ and $\chi[\phi/\psi]$ is a statement in which ψ has been replaced by ϕ (see [3], p. 35, for the use of the name of Frege and also [4] and [5]).

THEOREM 2. *In every Fregean structure if two objects are B-identical they are l-identical.*

PROOF. Straighforward by induction on the complexity of statements. □

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3.3. Extending the notion of morphism

Given a logical structure we define the notion of logical endomorphism as follows: $tv(\phi(\vec{a})) = tv(\phi(\varepsilon\ \vec{a}))$, where $\vec{a}$ is any sequence of objects of S (for example if $\vec{a} = <b, c>$, then $\phi(\varepsilon\ \vec{a}) = \phi(\varepsilon b, \varepsilon c)$). And we say that a logical structure is *logically rigid* i it has no proper logical endomorphism, (that is each logical endomorphism is 1–1).

   Notice that if ε is a logical endomorphism, a and εa are not necessarily logically equivalent, for example we can have, $tv(aRb) = 1$ and $tv(\varepsilon aRb) = 0$ and simultaneously $tv(aRb) = tv(\varepsilon aR\varepsilon b) = 1$.

THEOREM 3.   *The diagonal and logical identities coincide in a logical structure i it is logically rigid.*

RELATIONS BETWEEN
DIAGONAL IDENTITY, LOGICAL IDENTITY,
BOURBAKI INDENTITY
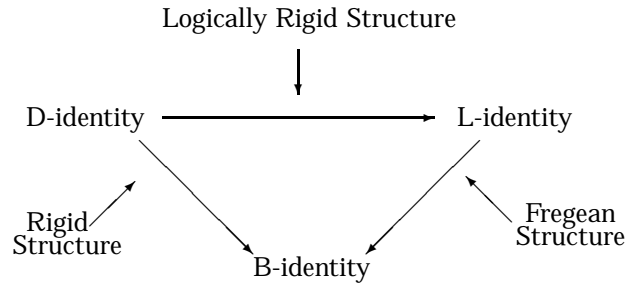
PROOF OF THEOREM 3.
($\Rightarrow$): Assume that in a logical structure the diagonal and logical identities coincide and suppose that for a logical endomorphism ε:
(1) $\varepsilon a = \varepsilon b$ and (2) a, b are not l-identical. From (2): $tv(\psi(a, c_1, \ldots, c_n)) = 1$ and $tv(\psi(b, c_1, \ldots, c_n) = 0$ for some ψ. Thus $tv(\psi(\varepsilon a, \varepsilon c_1, \ldots, \varepsilon c_n)) = 1$ and $tv(\psi(\varepsilon b, \varepsilon c_1, \ldots, \varepsilon c_n)) = 0$ which is impossible due to (1).
($\Leftarrow$): Suppose that di erent a and b are l-identical and consider the same function ε as in the proof ($\Rightarrow$) of Theorem 1. We want to show that ε is a

proper logical endomorphism. So consider any $x_1, \ldots, x_n \in S$ and a statement $\phi(x_1, \ldots, x_n)$. If $b \notin \{x_1, \ldots, x_n\}$ then obviously $tv(\phi(x_1, \ldots, x_n)) = tv(\phi(\varepsilon x_1, \ldots, \varepsilon x_n))$, and in case $b \in \{x_1, \ldots, x_n\}$, the same clearly follows due to the assumption that a, b are l-identical. $\quad\square$

References

[1] N. Bourbaki, *L'architecture des mathématiques (la mathématique ou les mathématiques?)*, [in:] **Les grand courants de la pensée mathématique**, edited by F. Le Lionnais, Cahiers du Sud, Paris, 1948, pp. 35–47.

[2] N. C. A. da Costa, *Ensaio sobre os fundamentos da lógica*, Hucitec, São Paulo, 1980.

[3] G. Frege, *Über Sinn und Bedeutung*, **Zeitschrift für Philosophie und philosophische Kritik** 100 (1992), pp. 25–50.

[4] D. Krause and J.-Y. Béziau, *Relativizations of the principle of identity*, **Journal of the Interest Group in Pure and Applied Logics** (to appear).

[5] R. Suszko, *Abolition of the Fregean axiom*, [in:] **Logic Colloquium**, edited by R. Parikh, Springer, Berlin, 1975, pp. 169–239.

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