Bulletin of the **S**ection of **L**ogic
Volume 28/2 (1999), pp. 89–97

Jean-Yves Béziau

A SEQUENT CALCULUS FOR LUKASIEWICZ'S THREE-VALUED LOGIC BASED ON SUSZKO'S BIVALENT SEMANTICS *

Abstract

A sequent calculus $S3$ for Lukasiewicz's logic $L3$ is presented. The completeness theorem is proved relatively to a bivalent semantics equivalent to the non truth-functional bivalent semantics for $L3$ proposed by Suszko in 1975. A distinguishing property of the approach proposed here is that we are keeping the format of the classical sequent calculus as much as possible.

Mathematics Subject Classification: 03B50, 03F03

1. Introduction

In this paper we present a sequent calculus $S3$ for Lukasiewicz's three-valued logic $L3$.

A few sequent calculi for $L3$ and other many-valued logics have already been proposed but generally they are modifications of the structure of Gentzen's original sequent calculi (see [14], [15], [11], [12], [3], [1], [2], [10]).

The system $L3$ presented here is quite orthodox: the sequents have the same structure as the sequents for classical logic and the structural rules are usual.

*This work was partially supported by the LOGIA project (Protem - CNPq)

$S3$ is closely related to Suszko's bivalent semantics for $L3$. In 1975 (cf. [13]), Suszko presented a bivalent semantics for $L3$. This may seem quite paradoxical unless one realized that this semantics is not truth-functional (for the discussion of this subject, see [4]).

$S3$ is based on the intuitive semantical interpretation of sequents and sequents rules. To prove the completeness of the system $S3$ relatively to Suszko's semantics we transform Suszko's conditions into an equivalent set of conditions, called tabular conditions, which can be read as the interpretations of sequents rules.

Originally Lukasiewicz (cf. [5]) conceived $L3$ as a set of tautologies. However later, various extensions of $L3$ to consequence relations were studied (cf. [7], [16]). Let us mention here two of them. The first is the extension to a "truth-preserving consequence" (cf. [17], p. 55). The other is the construction of systems of sequents for substructural consequence relations extending $L3$ (cf. [1]). The both solutions are more or less evident. So, e.g. Wójcicki claims that "There is no evidence that Lukasiewicz thought of the consequence operations characteristic of many-valued logic as truth-preserving ones. Rather he conceived them as determined by Modus Ponens." ([17], p. 279).

In the present paper, the system $S3$ is a sequent calculus for the logic $L3$ considered as a set of tautologies.

2. The sequent calculus $S3$

$S3$ has the same structural rules as the sequent calculus LK for classical logic (including the cut rule). We present $S3$ by means of finite sets of formulas denoted by Γ, Δ, etc. rather than sequences of formulas. Applying Gentzen's "$\longrightarrow$" for sequents, we represent a sequent as: $\Gamma \longrightarrow \Delta$.

We consider $L3$ only with implication ($\supset$) and negation ($\neg$). The (schemas of) logical rules of $S3$ are the following:

$$\frac{\longrightarrow a}{\neg a \longrightarrow}\ \neg l \qquad \frac{a \longrightarrow}{\neg\neg a \longrightarrow}\ \neg\neg l \qquad \frac{\longrightarrow a}{\longrightarrow \neg\neg a}\ \neg\neg r$$

$$\frac{\neg a \longrightarrow \qquad \longrightarrow \neg b}{a \supset b \longrightarrow}\ \supset l1 \qquad \frac{\longrightarrow a \qquad b \longrightarrow}{a \supset b \longrightarrow}\ \supset l2$$

$$\frac{\longrightarrow \neg a, b}{\longrightarrow a \supset b}\ \supset r1 \qquad \frac{a \longrightarrow \qquad \neg b \longrightarrow}{\longrightarrow a \supset b}\ \supset r2$$

$$\frac{a \supset b \longrightarrow \quad \longrightarrow a, \neg a \quad \longrightarrow b, \neg b}{\longrightarrow \neg(a \supset b)} \neg \supset r$$

$$\frac{a \supset b \longrightarrow \quad a \longrightarrow \quad \neg a \longrightarrow}{\neg(a \supset b) \longrightarrow} \neg \supset l1$$

$$\frac{a \supset b \longrightarrow \quad b \longrightarrow \quad \neg b \longrightarrow}{\neg(a \supset b) \longrightarrow} \neg \supset l2$$

For the sake of simplicity contexts are omitted here, however, let us emphasize that they work as in the standard sequent calculus for propositional classical logic.

$S3$ has the following property, which is close to the subformula property: the main formulas of the premisses of the rules are either proper subformulas of the main formula of the conclusion, or negations of its proper subformulas. It is easy to see that this property entails the decidability and the consistency of the system $S3$ without cut.

Someone well acquainted with the cut-elimination theorem can immediately see that cut-elimination holds for $L3$ (further details about this will be presented in another paper).

3. Suszko's bivalent semantics for $L3$ and its "tabular" version

In 1975, Suszko presented a non truth-functional bivalent semantics for $L3$. Such semantics is a set of functions ("bivaluations") from the set of formulas into the set $\{0, 1\}$, but these bivaluations are not homomorphisms between the syntactic algebra of formulas and a semantic algebra of truth-functions on $\{0, 1\}$, as in the classical case.

A bivaluation β of Suszko's semantics must obey the following conditions (cf. [13] or [8]):

<u>Suszko's conditions</u>
(a) $\beta(a) = 0$ or $\beta(\neg a) = 0$
(b) if $\beta(b) = 1$ then $\beta(a \supset b) = 1$
(c) if $\beta(a) = 1$ and $\beta(b) = 0$, then $\beta(a \supset b) = 0$
(d) if $\beta(a) = \beta(b)$ and $\beta(\neg a) = \beta(\neg b)$, then $\beta(a \supset b) = 1$
(e) if $\beta(a) = \beta(b) = 0$ and $\beta(\neg a) \neq \beta(\neg b)$, then $\beta(a \supset b) = \beta(\neg a)$

(f) if $\beta(\neg a) = 0$, then $\beta(\neg\neg a) = \beta(a)$
(g) if $\beta(a) = 1$ and $\beta(b) = 0$, then $\beta(\neg(a \supset b)) = \beta(\neg b)$
(h) if $\beta(a) = \beta(\neg a) = \beta(b)$ and $\beta(\neg b) = 1$, then $\beta(\neg(a \supset b)) = 0$.

Let us now consider the following set of conditions:
Tabular conditions
(1) if $\beta(a) = 0$ and $\beta(\neg a) = 0$ and $\beta(b) = 0$ and $\beta(\neg b) = 0$, then $\beta(a \supset b) = 1$
(2) if $\beta(a) = 0$ and $\beta(\neg a) = 0$ and $\beta(b) = 0$ and $\beta(\neg b) = 1$, then $\beta(a \supset b) = 0$
(3) if $\beta(a) = 0$ and $\beta(\neg a) = 0$ and $\beta(b) = 1$ and $\beta(\neg b) = 0$, then $\beta(a \supset b) = 1$
(4) if $\beta(a) = 0$ and $\beta(\neg a) = 1$ and $\beta(b) = 0$ and $\beta(\neg b) = 0$, then $\beta(a \supset b) = 1$
(5) if $\beta(a) = 0$ and $\beta(\neg a) = 1$ and $\beta(b) = 0$ and $\beta(\neg b) = 1$, then $\beta(a \supset b) = 1$
(6) if $\beta(a) = 0$ and $\beta(\neg a) = 1$ and $\beta(b) = 1$ and $\beta(\neg b) = 0$, then $\beta(a \supset b) = 1$
(7) if $\beta(a) = 1$ and $\beta(\neg a) = 0$ and $\beta(b) = 0$ and $\beta(\neg b) = 0$, then $\beta(a \supset b) = 0$
(8) if $\beta(a) = 1$ and $\beta(\neg a) = 0$ and $\beta(b) = 0$ and $\beta(\neg b) = 1$, then $\beta(a \supset b) = 0$
(9) if $\beta(a) = 1$ and $\beta(\neg a) = 0$ and $\beta(b) = 1$ and $\beta(\neg b) = 0$, then $\beta(a \supset b) = 1$
(10) if $\beta(a) = 1$, then $\beta(\neg a) = 0$
(11) if $\beta(a) = 0$, then $\beta(\neg\neg a) = 0$
(12) if $\beta(a) = 1$, then $\beta(\neg\neg a) = 1$
(13) if $\beta(a \supset b) = 0$ and $[\beta(a) = 1$ or $\beta(\neg a) = 1]$ and $[\beta(b) = 1$ or $\beta(\neg b) = 1]$, then $\beta(\neg(a \supset b)) = 1$
(14) if $\beta(a \supset b) = 0$, and $\beta(a) = 0$ and $\beta(\neg a) = 0$ then $\beta(\neg(a \supset b)) = 0$
(15) if $\beta(a \supset b) = 0$, and $\beta(b) = 0$ and $\beta(\neg b) = 0$ then $\beta(\neg(a \supset b)) = 0$.

Theorem *Suszko's conditions are equivalent to tabular conditions.*

Proof.

I. Suszko's conditions imply tabular conditions

It is easy to check that:

(1) and (5) follow from (d);

(2) and (4) from (e);

(3), (6) and (9) from (b);

(7) and (8) from (c);

(10) from (a);

(11) is a consequence of (a) and (f);

(12) is a consequence of (a) and (f);

(14) is a consequence of (h).

Now let us examine the case of (13) and (15):

(13) is a consequence of (g) and (5), (6), (10). We use the fact that (5), (6) and (10) are consequences of (d), (b) and (a).

By (g), if $\beta(a) = 1$ and $\beta(\neg a) = 0$ and $\beta(b) = 0$ and $\beta(\neg b) = 1$, then $\beta(\neg(a \supset b)) = 1$.

So we have to show that if [$\beta(a \supset b) = 0$, and $\beta(a) = 1$ or $\beta(\neg a) = 1$, and $\beta(b) = 1$ or $\beta(\neg b) = 1$], then [$\beta(a) = 1$ and $\beta(\neg a) = 0$ and $\beta(b) = 0$ and $\beta(\neg b) = 1$].

Suppose that $\beta(a \supset b) = 0$, and $\beta(a) = 1$ or $\beta(\neg a) = 1$, and $\beta(b) = 1$ or $\beta(\neg b) = 1$.

Suppose that $\beta(a) = 0$, then $\beta(\neg a) = 1$, due to the above supposition. If $\beta(b) = 0$, then, by (10), $\beta(\neg b) = 1$; applying (5), we get $\beta(a \supset b) = 1$, which is absurd. If $\beta(b) = 1$, $\beta(\neg b) = 0$; applying (6), we get $\beta(a \supset b) = 1$, which is absurd. Consequently $\beta(a) = 1$ and $\beta(\neg a) = 0$.

Suppose that $\beta(b) = 1$, then by (10), $\beta(\neg b) = 0$; applying (9), we get $\beta(a \supset b) = 1$, which is absurd. Consequently $\beta(b) = 0$ and $\beta(\neg b) = 1$.

(15) is a consequence of (g) and (1), (4), (10). We use the fact that (1), (4) and (10) are consequences of (d), (e) and (a).

By (g), if $\beta(a) = 1$ and $\beta(\neg a) = 0$ and $\beta(b) = 0$ and $\beta(\neg b) = 0$, then $\beta(\neg(a \supset b)) = 0$.

So we have to show that if [$\beta(a \supset b) = 0$, and $\beta(b) = 0$ and $\beta(\neg b) = 0$] then, [$\beta(a) = 1$ and $\beta(\neg a) = 0$.

Suppose that $\beta(a \supset b) = 0$ and $\beta(b) = 0$ and $\beta(\neg b) = 0$.

Suppose that $\beta(a) = 0$ and $\beta(\neg a) = 0$, then by (1), $\beta(a \supset b) = 1$, which is absurd.

Suppose that $\beta(a) = 0$ and $\beta(\neg a) = 1$, then by (4), $\beta(a \supset b) = 1$, which is absurd.

II. Tabular conditions imply Suszko's conditions.
It is easy to check that:
(a) follows from (10);
(b) is a consequence of (3), (6), (9) and (10);
(c) is a consequence of (7), (8) and (10);
(d) is a consequence of (1), (5), (9) and (10);
(e) is a consequence of (2) and (4);
(f) is a consequence of (11) and (12);
(h) is a consequence of (2) and (14).

Now let us examine the case of (g):
(g) is a consequence of (13), (15), (7), (8) and (10).
Suppose $\beta(a) = 1$ and $\beta(b) = 0$, then by (10), $\beta(\neg a) = 0$.
If $\beta(\neg b) = 0$, by (7) and (15), $\beta(\neg(a \supset b)) = 0$, therefore $\beta(\neg(a \supset b)) = \beta(\neg b)$.
If $\beta(\neg b) = 1$, by (8) and (13), $\beta(\neg(a \supset b)) = 1$, therefore $\beta(\neg(a \supset b)) = \beta(\neg b)$.

4. Proof of the completeness theorem

With the system $S3$, we define in a standard way a binary relation $\vdash$ between sets of formulas and formulas, i.e. $T \vdash x$ i there is a finite subset Γ of T such that the sequent $\Gamma \longrightarrow x$ is derivable in $S3$.

The semantic relation $\models$ is defined with Suszko's conditions, i.e. $T \models x$ i for every bivaluation β, if $\beta(y) = 1$ for every formula y of T, then $\beta(x) = 1$.

Theorem (Completeness)

$$\textit{If } T \nvdash x \textit{ then } T \not\models x$$

Proof. Due to Lindenbaum-Asser theorem, we know that if $T \nvdash x$, there is a theory V which is an extension of T and which is relatively maximal in x, that is to say, $V \nvdash x$ and for every formula b not in V, $V \cup \{b\} \vdash x$. If we succeed to show that the characteristic function v of V obeys the fifteen tabular conditions equivalent to Suszko's conditions, then this will be enough to show that $T \not\models x$.

(For these facts consult [9] or [17].)

First note that $V \nvdash a$ i $a \notin V$ i $v(a) = 0$.

Suppose $v(a) = 0$ and $v(\neg b) = 0$, we show that $v(a \supset b) = 1$, therefore that conditions (1), (3), (4) and (6) are satisfied.

If $v(a) = 0$ and $v(\neg b) = 0$, then $V \nvdash a$ and $V \nvdash \neg b$, therefore $V, a \vdash x$ and $V, \neg b \vdash x$. It means that there are finite subtheories Γ and Δ of V such that the sequents Γ, a $\longrightarrow x$ and $\Delta, \neg b \longrightarrow x$ are derivable in $S3$. Applying the rule $\supset r2$, we see that the sequent $\Gamma, \Delta \longrightarrow a \supset b, x$ is derivable in S3. Now suppose that $v(a \supset b) = 0$. This means that there is a sequent $\Sigma, a \supset b \longrightarrow x$ (where Σ is a finite subset of V), which is derivable in $S3$. Applying the cut rule, we see that the sequent $\Gamma, \Delta, \Sigma \longrightarrow x$ is derivable in $S3$ and that therefore $V \vdash x$, which is absurd.

Suppose $v(\neg a) = 1$ or $v(b) = 1$, we show that $v(a \supset b) = 1$, therefore that conditions (3), (4), (5), (6) and (9) are satisfied.

The proof is similar using the rule $\supset r1$.

Suppose $v(\neg a) = 0$ and $v(\neg b) = 1$, we show that $v(a \supset b) = 0$, therefore that conditions (2) and (8) are satisfied.

The proof is similar using the rule $\supset l1$.

Suppose $v(a) = 1$ and $v(b) = 0$, we show that $v(a \supset b) = 0$, therefore that conditions (7) and (8) are satisfied.

The proof is similar using the rule $\supset l2$.

We show, in a similar way, that v obeys the conditions (10), (11), (12), (13), (14) and (15) applying the rules $\neg l$, $\neg\neg l$, $\neg\neg r$, $\neg \supset r$, $\neg \supset l1$, $\neg \supset l2$, respectively.

Proposition (Soundness)

$$\textit{If } T \vdash x \textit{ then } T \models x$$

Proof. Standard straightforward.

Acknowledgement. I would like to thank G. Malinowski and J. Zygmunt as well as two anonymous referees for various useful comments on a previous draft of this paper.

References

[1] A. Avron, *Natural 3-valued logics - Characterization and proof theory*, Journal of Symbolic Logic 56 (1991), pp. 276–294.

[2] A. Avron, *The method of hypersequents in the proof theory of propositional non classical logic*, [in:] Logic : from foundations to applications, W. Hodges *et al.* (eds), Oxford University Press, Oxford, 1996, pp. 1–32.

[3] W. A. Carnielli, *On sequents and tableaux for many-valued logics*, Journal of Non-Classical Logic 8 (1991), pp. 59–76.

[4] N. C. A. da Costa, J.-Y. Béziau and O. A. S. Bueno, *Malinowski and Suszko on many-valued logics: on the reduction of many-valuedness to two-valuedness*, Modern Logic 6 (1996), pp. 272–299.

[5] J. Lukasiewicz, *O logice trójwartościowej*, Ruch Filozoficzny 5 (1920), pp. 170–171. English translation in [6], pp. 87–88.

[6] J.Lukasiewicz - Selected works, L.Borkowski (ed), North-Holland, Amsterdam, 1970.

[7] G. Malinowski, *Historical note*, in [18], 1977, pp. 177–199.

[8] G. Malinowski, Many-valued logics, Oxford University Press, Oxford, 1993.

[9] W. A. Pogorzelski and P. Wojtylak, Elements of the theory of completeness in propositional logic, Silesian University, Katowice, 1982.

[10] A. Prijatelj, *Bounded contraction and Gentzen-style formulation of Łukasiewicz logics*, Studia Logica 57 (1996), pp. 437–456.

[11] G. Rousseau, *Sequents in many-valued logic I*, Fundamenta Mathematicae 60 (1967), pp. 23–33.

[12] G. Rousseau, *Sequents in many-valued logic II*, Fundamenta Mathematicae 67 (1970), pp. 125–131.

[13] R. Suszko, *Remarks on Lukasiewicz's three-valued logic*, Bulletin of the Section of Logic 4 (1975), pp. 87–90.

[14] M. Takahashi, *Many-valued logics of extended Gentzen style I*, Science Reports of the Tokyo Kyoiku Daigaku. Section A 9 (1968), pp. 271–292.

[15] M. Takahashi, *Many-valued logics of extended Gentzen style II*, Journal of Symbolic Logic 35 (1970), pp. 493–528.

[16] R. Wójcicki, *On matrix representations of consequence operations of Łukasiewicz's sentential calculi*, Zeitschrift für matematische Logik und Grundlagen der Mathematik 19 (1973), pp. 239–247.

[17] R. Wójcicki, Theory of logical calculi. Basic theory of consequence operations, Kluwer, Dordrecht, 1988.

[18] R. Wójcicki and G. Malinowski, Selected papers on Lukasiewicz sentential calculi, Polish Academy of Sciences, Warsaw, 1977.

LNCC/CNPq
National Laboratory for Scientific Computing
Avenida Getulio Vargas 333
25651-070 Petrópolis, RJ, Brazil
jyb@lncc.br