A Paradox in the Combination of Logics

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In this paper we present a fact, surprising enough to be called a paradox, which shows that the central issue in combination of logic is still problematic. This issue has been described by Dov Gabbay in his book on fibration as follows "Combine $S1$ and $S2$ into a system S which is the smallest logical system for the combined language which is a conservative extension of both $S1$ and $S2$. The two systems are presented in totally different ways. How are we going to combine them." ([2], p.7)

Given two logics $L1$ and $L2$, let us call $L1*L2$ the combination of $L1$ and $L2$ described by Gabbay, i.e. the smallest logic for the combined language which is a conservative extension of both $L1$ and $L2$. If we have a mechanism for combining semantics or proof systems, how can we be sure that this mechanism produces $L1 * L2$? If we have a technique to combine a Kripke semantics $K1$ generating a logic $L1$ and a Kripke semantics $K2$ generating a logic $L2$, we would like to be sure that the combination of $K1$ and $K2$ generates the combined logic $L1 * L2$. Modal logic is one of the favourite subject of logic combinators and it has been investigated since many years, so it is not surprising that people have found some techniques producing the expected result. But there are some other cases, where there is not yet a solution. The difficulty does not appear in a remote region of the logic land, e.g. the combination of super turbo polar fuzzy logics, but in a very simple case: good old classical propositional logic.

Consider the semantics SC for classical conjunction, given by the following usual condition: $b(F \wedge G) = 1$ iff $b(F) = 1$ and $b(G) = 1$. We call LC the consequence relation (logic, for short) generated by this condition using the usual method.

Similarly we consider the semantics SD for classical disjunction, given by the following usual condition: $b(F \vee G) = 1$ iff $b(F) = 1$ or $b(G) = 1$ and we call LD the generated logic.

Now if we put together the two conditions SC and SD in the natural way, we get a logic LCD which is not the expected one, it is not $LC * LD$, i.e. the smallest logic for the combined language which is a conservative extension of both LC and LD.

In the logic LCD generated by the combination of SC and SD, we have distributivity between conjunction and disjunction:

$(F \wedge G) \vee H \dashv\vdash (F \vee H) \wedge (G \vee H)$

$(F \vee G) \wedge H \dashv\vdash (F \wedge H) \vee (G \wedge H)$

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The reader can check this with the truth-table method. But distributivity does not hold in $LC * LD$ by definition. Strangely enough, the combination of SC and SD produces something new, which was apparently neither in SC nor in SD. This kind of combination remembers biological phenomena and should perhaps better be called copulation. Note furthermore that here we have two logics which are presented in a very similar way, not heterogeneous presentations as suggested by Gabbay. So the challenge seems bigger than expected.

What can be said is that truth-functionality is not preserved by combination, since LC and LD are truth-functional (i.e. have a truth-functional semantics) but not $LC * LD$. The combination of SC and SD is a particular case of combination of logical matrices. What the paradox shows is that if we combine logical matrices in the natural way, we don't necessarily get what we want.

We find a similar problem at a proof-theoretical level. Let us consider the system GC which is the Gentzen system that we get by keeping only the structural rules and the two rules for conjunction of Gentzen's sequent system LK for classical logic. We consider in a similar way the system GD. Now the system GCD that we get by putting together the rules of GC and GD generates the logic LCD. In particular it is sound and complete for the combination of the semantics SC and SD (see [1]). It is in fact easy to prove distributivity in this system. Distributivity appears as derived rules. What shows the paradox here, is that if we put rules of two systems together, we may get more than expected, like if the rules were copulating.

It is possible to find a Gentzen sequent system for $LC * LD$. Consider the system $GC1$ which is the same as GC except that sequents must have one and only one formula on both sides. We define $GD1$ in a similar way. $GC1$ and $GD1$ generates respectively LC and LD, and when we put them together we get a system which generates $LC * KD$. They do not copulate, or if they do, this does not produce a fruit.

We may find several other examples where this kind of paradox appears. The paradox can be explained by the fact that in a semantics, or in a proof system, all features are not explicit. The implicit features may not manifest themselves isolately, but they may manifest, become active and produce something new by the combination process. Combination then turns into productive copulation.

The same logic can be generated by many different methods. But the fact that different methods generate the same logic does not mean that these methods are equivalent in general. LC can be generated by a standard system of sequents, namely GC, and by a substructural one, $GC1$. This doesn't mean that substructural and non substructural systems of sequents are the same.

References

[1] J.-Y.Béziau, 1995, *Recherches sur la Logique Universelle*, Department of Mathematics, University of Paris 7, Paris.

[2] D.Gabbay, 1999, Fibring logics, Clarendon, Oxford, 1999.

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