Classical Negation can be Expressed by One of its Halves

JEAN-YVES BÉZIAU, *National Laboratory for Scientific Computing, Avenida Getúlio Vargas 333, 25651-070 Petrópolis, RJ, Brazil.*
E-mail: jyb@lncc.br

Abstract

We present the logic $K/2$ which is a logic with classical implication and only the left part of classical negation.

We show that it is possible to define a classical negation into $K/2$ and that the classical propositional logic K can be translated into this apparently weaker logic.

We use concepts from model-theory in order to characterized rigorously this translation and to understand this paradox. Finally we point out that $K/2$ appears, following Haack's distinction, both as a deviation and an extension of K.[1]

Keywords: translation, deviant logic, extension of classical logic

1 Introduction

Consider the well-known two classical semantic conditions for implication and negation:

$$\beta(a \to b) = 0 \text{ iff } \beta(a) = 1 \text{ and } \beta(b) = 0$$

$$\beta(a) = 1 \text{ iff } \beta(\neg a) = 0.$$

Due to the fact that other classical connectives are definable in terms of implication and negation, these two conditions define the whole classical propositional logic.

Now imagine that instead of the above condition for classical negation, we take only the following "half" condition:

$$\text{if } \beta(a) = 1 \text{ then } \beta(\neg a) = 0.$$

In this paper we will show that although the logic $K/2$, defined by this condition together with the condition for implication, is clearly weaker than the classical propositional logic K, it is possible to translate K into $K/2$.

When we substitute the half condition for the full condition for negation, we are defining a bivalent semantics which is quite different from the usual one. This new semantics is not truth-functional in the sense that bivaluations are not homomorphims and in particular a distribution on atomic formulas does not uniquely extend to a bivaluation of the semantics.

[1] Full version of a contributed paper presented at the *4th Workshop on Logic, Language, Information and Computation* (*WoLLIC'97*), `http://www.di.ufpe.br/~wollic97`, held in Fortaleza (Ceará), Brazil, August 19–22 1997, with scientific sponsorship by IGPL, FoLLI, and ASL, and organised by Univ. Federal do Ceará (UFC) and Univ. Federal de Pernambuco (UFPE).

Anyway, bivalent non truth-functional semantics allow to define logics in a quite natural way. R. Suszko constructed such a bivalent semantics for Łukasiewicz's three valued logic (see [10]) and a systematic study of such semantics has been developed by N. C. A. da Costa (see e.g. [4]).

Moreover in [1] we have presented a technique which establishes a tight relation between bivaluations and rules of sequent rules in such a way that, for example, it is possible to see immediately that the sequent calculus with the standard structural rules (weakening, permutation, contraction, identity and cut), the two rules for classical implication and only the left rule for negation is a sound and complete axiomatization of $K/2$.

A brief analysis of the standard proof of the cut-elimination's theorem shows that this proof can be adapted without problem to the case of this sequent calculus for $K/2$, according to some remarks of Raggio for a sequent calculus similar to this one (see [9]).

Our main preoccupation in this paper is not to present these technical results but to analyse the relation between K and $K/2$.

2 The logic $K/2$

We consider a set of formulas $\mathcal{F}$ constructed with one binary connective $\succ$ and one unary connective $\ominus$.

We define a set $\mathcal{B}$ of functions from $\mathcal{F}$ to $\{0, 1\}$ as follows: Given any two formulas a and b, $\beta \in \mathcal{B}$ iff

$$\beta(a \succ b) = 0 \text{ iff } \beta(a) = 1 \text{ and } \beta(b) = 0$$

$$\text{if } \beta(a) = 1 \text{ then } \beta(\ominus a) = 0.$$

With this set of bivaluations, we define a consequence relation $\models_{K/2}$ in the usual way: Given any formula a and set of formulas T,

$T \models_{K/2} a$ iff for every $\beta \in \mathcal{B}$, if $\beta(b) = 1$ for all $b \in T$ then $\beta(a) = 1$.

The logic $K/2$ is the following structure:

$$K/2 = \langle \mathcal{F}; \succ; \ominus; \models_{\mathcal{K}/\in} \rangle.$$

Consider an atomic formula a and the function f such that $f(a) = 0$, $f(\ominus a) = 0$, $f(\ominus a \succ a) = 1$, $f((\ominus a \succ a) \succ a) = 0$.

It is possible to prove that there is an extension of f which is a member of $\mathcal{B}$ and that therefore $\not\models_{K/2} (\ominus a \succ a) \succ a$.

This is quite obvious. However to prove this rigorously one has to follow a general method presented in [3] or [4].

Let us say that two formulas a and b are logically equivalent iff $a \models_{K/2} b$ and $b \models_{K/2} a$.

We have the following result:

$a \succ (b \succ c)$ and $b \succ (a \succ c)$ are *logically equivalent* but not $\ominus(a \succ (b \succ c))$ and $\ominus(b \succ (a \succ c))$.

This means that the replacement theorem does not hold for $\ominus$.

3 Inclusion of $K/2$ into K

We consider classical propositional logic with only implication $\to$ and negation $\neg$. It is the following structure:

$$K = \langle \mathcal{K}; \to; \neg; \models_{\mathcal{K}} \rangle.$$

Consider a bijection φ between the atomic formulas of $\mathcal{F}$ and the atomic formulas of $\mathcal{K}$, as $\langle \mathcal{F}; \succ; \ominus \rangle$ and $\langle \mathcal{G}; \to; \neg \rangle$ are absolute free algebras of the same type, there is a unique extension ι of φ which is an isomorphism between these two algebras, i.e.:
$\iota(a \succ b) = \iota(a) \to \iota(b)$
$\iota(\ominus a) = \neg(\iota(a))$.
We will call such an isomorphism a *language-isomorphism*.

It is clear that we have:

$$\text{if } T \models_{K/2} a \text{ then } \iota(T) \models_K \iota(a)$$

but not the converse, because as we have seen, $\not\models_{K/2} (\ominus a \succ a) \succ a$ and as it is known $\models_K (\neg a \to a) \to a$.

To simplify the matter, we could have taken for $K/2$ the same set of formulas as for K. In this case the function ι could have been the identity function.

In this case $K/2$ is strictly included in K in the sense that the relation $\models_{K/2}$ is strictly included in the relation $\models_K$. Therefore it seems that we can say that $K/2$ is strictly weaker than K.

Taking different sets of formulas does not change the fact. We can say that $K/2$ is strictly included, up to language-isomorphism, in K.

One might want to interpret this fact saying that $K/2$ is a proper sublogic of K or that K is a strict extension of $K/2$. However one must be careful, because if this shall mean that $K/2$ is a proper substructure of K or that K is a structure which is a proper extension of $K/2$, the words "substructure" and "extension" are not used in the usual manner.

What is the usual manner? Here is an example: $\langle \mathbb{N}; +; \times; \leq \rangle$ is a proper substructure of $\langle \mathbb{Z}; +; \times; \leq \rangle$ (and the latter is a proper extension of the former). To extend a structure means that the domain of the structure is extended and that the functions and the relations are extended to the new domain, staying invariant on the original domain.

This is not what happens here. The domain does not change, but the relation of consequence is extended.

$K/2 = \langle \mathcal{F}; \succ; \ominus; \models_{\mathcal{K}/\in} \rangle$ and $K = \langle \mathcal{K}; \to; \neg; \models_{\mathcal{K}} \rangle$ are not first-order structures, because the consequences relations are binary relations between sets of objects of the domains of the structures and objects of the domains of the structures. We could have considered the case where the symbol "$\models$" denotes only a monadic predicate defining a set of tautologies rather than a consequence relation.

Anyway, standard definitions of first-order structures can be applied in a straightforward manner to these second-order structures. For example a *homomorphism* μ from $K/2$ into K is a function from $\mathcal{F}$ into $\mathcal{K}$ such that:
(1) $\mu(a \succ b) = \mu(a) \succ \mu(b)$
(2) $\mu(\ominus a) = \neg\mu(a)$

(3) $T \models_{K/2} a$ iff $\mu(T) \models_K \mu(a)$.
If μ is a bijection, μ is said to be an *isomorphism*.

According to this definition, the function ι above is not a homomorphism from $K/2$ into K, because the condition (3) is not satisfied.

The conclusion of this section is that given the standard definitions, we cannot say, in view of the relation established between $K/2$ and K by the function ι, that $K/2$ is a substructure of K or that $K/2$ is embedded into K.

Hence, let us simply say that $K/2$ is strictly included (up to language-isomorphism) into K.

This relation of inclusion is well-known in logic: intuitionistic logic, Łukasiewicz's three-valued logic, positive classical logic (i.e. without negation) are strictly included into classical logic.

4 A translation of K into $K/2$

4.1 Definition of the translation

It can easily be proved that for every β of B:

$$\beta(a) = 1 \text{ iff } \beta(a \succ \ominus a) = 0.$$

Consider the function τ from $\mathcal{K}$ into $\mathcal{F}$, such that:
$\tau(a) = a$, if a is atomic
$\tau(a \to b) = \tau(a) \succ \tau(b)$
$\tau(\neg a) = \tau(a) \succ \ominus(\tau a)$.
It can be proved that

$$T \models_K a \text{ iff } \tau(T) \models_{K/2} \tau(a).$$

One can interpret this result saying that K is translatable into $K/2$. This suggests that $K/2$ is at least as strong as K. We will see in fact that $K/2$ cannot be translated in a similar way into K. In view of the strict inclusion of $K/2$ into K this seems paradoxical.

A situation quite similar happens with intuitionistic logic. Gödel has even shown that it is possible to translate Peano's classical arithmetic into Heyting's intuitionistic arithmetic in such a way that the inconsistency of the former would entail the inconsistency of the latter.

Even if nowadays the various translations of classical logic into intuitionistic logic are well-known, the fact that classical logic can be translated into intuitionistic logic, which is strictly included into it, is still a paradox because it is against intuition and has not yet been properly explained. (On this subject see [8]).

The case of $K/2$ is a similar paradox. One can say that it is even more striking, in view of the simplicity of $K/2$.

4.2 Mathematical characterization of the translation

The function τ is not a morphism of K into $K/2$ because it does not respect the corresponding condition (2') of the definition of morphism given in Section 3:

(2') $\mu(\neg a) = \ominus\mu(a)$.

And one may rightly think that, if we consider as translations between logics functions just respecting the consequence relation and not the connectives, it is easy to translate classical logic in a logic strictly included in it.

However the function τ has some additional properties which can rightly allow one to say that it must be seriously considered.

Note that this kind of function is similar to the one we use when, in classical logic, we define conjunction in terms of negation and implication and which allows one to say that classical logic with conjunction, negation and implication is translatable into classical logic with negation and implication only.

Let us see now how this function can be mathematically characterized considering the well-known notions of model theory of expansion (reduct) and definitional expansion (see e.g. [7]).

We consider the following definition:

$$\sim a =_{Def} a \succ \ominus a$$

and the definitional expansion $EK/2$ of the structure $K/2$:

$$EK/2 = \langle E; \succ; \ominus; \sim; \models_{EK/2} \rangle.$$

Now the function τ above can be seen as an isomorphism between classical logic K and the following reduct $REK/2$ of $EK/2$:

$$REK/2 = \langle E; \succ; \neg; \models_{EK/2} \rangle.$$

Note that $EK/2$ cannot be considered as a definitional expansion of $REK/2$, because it is not possible to define $\ominus$ with $\succ$ and $\sim$. The basic reason is that the replacement theorem does not hold for $\ominus$.

If we say that two structures are *equivalent* iff they have a common (up to isomorphism) definitional expansion, $EK/2$ is equivalent to $K/2$ but $REK/2$ is not equivalent to them.

Now, if we say that a *strict reduct* is a reduct of a structure not equivalent to it, we can summarize the situation as follows:

Classical logic K is isomorphic to a strict reduct of a structure equivalent to $K/2$.

5 Open problems and discussion

Embeddable means isomorphic to a substructure. How to interpret the difference between strict embedding (embedding in a proper substructure) and the situation described above? Would it be correct to widen the notion of embedding and to say that K is embeddable into $K/2$? Or can we find an example, in logic, where these two notions radically differ?

Just let us say here that a logic $L1$ is (*strictly*) *translatable* into a logic $L2$ iff $L1$ is (strictly) embeddable into $L2$ (up to equivalence) or $L1$ is isomorphic to a strict reduct of $L2$ (up to equivalence). (compare with [2] [5]).

What we call the translation paradox, is, given two logics $L1$ and $L2$, the conjunction of the following facts:

- $L1$ is strictly included (up to language-isomorphism) into $L2$
- $L2$ is translatable into $L1$.

This remembers the Galilean paradox, i.e. the fact that the set of even numbers is strictly included into the set of natural numbers and that at the same time there is a one-to-one correspondance between these two sets. One can say that this paradox was solved by the framework of set-theory which perfectly explains this difference.

To solve the translation paradox, we must find a framework according to which inclusion and translatibility between logics have intuitive interpretations which are not incompatible.

<u>Conditions of translability</u>

In order to study this problem, it will be useful to study many examples of this paradox, and in particular to find some conditions which allow to translate classical logic into a logic strictly included in it.

This question can in turn be examined by trying to solve the following problem:

<u>The open problem of a minimal logic</u>

Find a logic M such that
- M is strictly included into K
- K is translatable into M
- there is no logic strictly included into M such that K is translatable in it.

<u>Haack's distinction is paradoxical</u>

Finally let us note that discussions about comparisons of logic have been carried out in a rather informal way by philosophers of logic. For example S.Haack (cf. [6]) says that a *deviant* logic is a logic which has the same language as classical logic but not the same set of theorems or consequence relation (example: Łukasiewicz's three-valued logic $L3$) and that an *extension* of classical logic is a logic whose language is an extension of the language of classical logic (example: the modal logic $S4$). According to her, an example of logic which is both a deviation and an extension of classical logic would be a modal extension of Łukasiewicz's three-valued logic.

However these definitions are not very rigorous and it would be better to found the whole discussion on mathematical definitions considering logics as structures and using concepts from model-theory in order to compare these structures. This is what we have tried to do here in the case of $K/2$.

Note that, following the standard definitions of model theory, $S4$ appears rather as an *expansion* of classical logic than as an extension.

The logic $K/2$, at firt sight, is a deviant logic, according to Haack's definition, but its definitional expansion $EK/2$ is an expansion of K. $EK/2$ stands with regards to K exactly as $S4$ stands with regards to K. Therefore, following Haack's terminology, $EK/2$ is an extension of K. As $K/2$ is equivalent to $EK/2$, we can say that $K/2$ is both a deviation and an extension of K. But this example is quite different from Haack's kind of examples because K is not translatable into such a logic.

Haack's distinction therefore is quite confuse and does not seem a good framework to explain the translation paradox.

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