What is the principle of identity?
(identity, congruence and logic)

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Abstract: The principle of identity has different formulations. Some, given by philosophers, are pure nonsense. Even logicians make frequent mistakes. In first-order logic, the so-called Leibniz scheme, does not define identity but the strongest congruence relation. On the other hand, the weakest congruence relation, trivial identity or diagonal identity, is not axiomatizable in first-order model theory, in the same sense that the notion of well-order is not axiomatizable; and despite the completeness of first-order logic with identity, when identity is taken as primitive.

> "Identity is the strongest of the equivalent relations that are transitive, reflexive and symmetrical." (Ruth Barcan-Marcus, 1993, p. 202).

1. Introduction

The principle of identity has been considered in the philosophical tradition as one of the fundamental principles of logic and more generally as one of the fundamental principles of thought, together with the principles of contradiction and excluded middle. These two last principles have received formulations in the context of modern logic which are relatively clear. It seems that it is not the case of the principle of identity.

There are several non equivalent formulations of this principle. One can get a good understanding of these different formulations through the mathematical concept of congruence relation. The

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simplest form of the principle of identity, trivial identity, is not axiomatizable in a model-theoretical sense in first-order logic.

2. Reflexivity and trivial identity

Many philosophers consider the following statement as the principle of identity:

(RI) *Every object is identical to itself* or *Everything is identical to itself.* They also express it more formally by:

(FRI) *For every a, a=a*

However it is clear that such kind of formulations are false or in the best case highly ambiguous. If one agrees that (FRI) is the correct formalization of (RI), then (RI) just means that the identity is a binary *reflexive* relation between things or objects. But there are a lot of reflexive relations which cannot be considered as identity relations. Reflexivity is a necessary condition for identity but not at all a sufficient condition.

According to A. Stroll (1967, p. 122), "Almost all writers (...) from Descartes to Kant, took the term 'identity' to mean that an object 'is the same with itself' (Hume). These formulations were expressed by the logical principle, regarded as one of the basic laws of reasoning, $[(x), x=x]$."[2]

It is strange that so much great philosophers have made such a confusion. Maybe what they wanted to say is:

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[2] The formula "$(x), x=x$" here is obviously an anachronism. The universal quantifier was not formalized by people like Descartes or Kant. The closer they get at it is (FRI).

(TI) *Every object is identical to itself and different from all other objects.*

Everyone understands the intuitive idea of identity expressed by (TI). Let us call it *trivial identity*. The idea of trivial identity can be formulated in various different manners: “Anything whatsoever has the relation of identity to itself, and to nothing else” (Williamson, 1998, p. 675) or “identity, the relation each thing bears just to itself” (Wagner, 1995, p. 358). All these formulations hide the tautological feature of the definition, which appears clearly if we reformulate (TI) as follows:

(TII) *Every object is identical to itself and different from different objects.*

We can furthermore replace “different” by “not identical” and we get:

(TIII) *Every object is identical to itself and not identical to non identical objects.*

In first-order logic, this corresponds to the formula:

$$\forall x\,(x{=}x \wedge \forall y\,(x \neq y \rightarrow x \neq y))$$

This formula is equivalent to the formula

$$\forall x\,(x{=}x)$$

and so (TI) is in fact just (RI).

One can wonder if it is possible at all to correctly express what we have called trivial identity. If for example there is a formalization of trivial identity in first-order logic or second-order logic.

Wagner (1995, p. 358) writes: “Identity, the relation each thing bears just to itself. Formally, $a = b \leftrightarrow \forall F\,(Fa \rightarrow Fb)$.” Is this formula, or better the formula $\forall x \forall y\,(x=y \leftrightarrow \forall F(Fx \rightarrow Fy))$, a correct formalization of trivial identity in second-order logic? Is the scheme of formula $\forall x\, \forall y\,(x=y \leftrightarrow (\varphi x \rightarrow \varphi y))$ a correct formalization of trivial identity in first-order logic?

3. Congruence

Some philosophers make a more subtle confusion, they think that identity is just an *equivalence relation*, i.e. a relation which is reflexive, symmetrical and transitive. But once again this is a necessary condition but in general not a sufficient condition. The relation must be moreover compatible. This compatibility condition is in fact the key to identity and corresponds to the concept of congruence.

This concept is a mathematical concept which has arisen from some particular mathematical theories (geometry, algebra, etc.). However a general definition of it can be given in the context of mathematics considered as a theory of structures, following the spirit of modern mathematics. Such a general definition was given by Bourbaki.

The notion of congruence can be defined for any kind of structures. Here we will focus on very simple structures of type $<A; R>$ where A is a set and R is a binary relation on A. One can get a general definition of congruence just as a straightforward definition of this particular example.

In the context of a given structure $<A; R>$, we define a congruence relation $\cong$ as a binary relation which is an equivalence relation and which moreover obeys the following *condition of compatibility:*

$$\text{(C)}\ a \cong b \Rightarrow R[a, c] \textit{ iff } R[b, c] \leftrightarrow .$$

The notation "$R[a, c]$" is an abbreviation for aRc or cRa and the second part of the conditional should be understood as: for any c, aRc iff bRc, and *cRa* iff *cRb*. One can interpret informally this condition by "if two objects are congruent they have the same position in the structure".[3]

4. Diagonal, trivial congruence and trivial identity

Given a structure of type $<A; R>$, we can consider the *diagonal* of the cartesian product *AXA*. This notion is familiar to any working mathematician. For example if the set *A* has three elements *a*, *b*, *c*, the diagonal of *AXA* is the set with the elements (*a*; *a*), (*b*; *b*), (*c*; *c*).

Obviously the diagonal of *AXA* corresponds to the notion of trivial identity. One may think that the working mathematician has therefore a solution to the problem of defining trivial identity. But one must be careful about naïve mathematical terminology. It is not clear at all how the working mathematician defines precisely the diagonal.

The diagonal of *AXA* is a congruence relation, it is the weakest one. The relations of congruence of a structure of type $<A; R>$ form a lattice and the bottom of the lattice is the diagonal of *AXA*. In fact a correct way to define the diagonal of *AXA* is to define it as the bottom of the lattice of congruences. We will call such bottom, the *trivial congruence*. Trivial congruence is a mathematical formulation of trivial identity.

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[3] As it is known, there is a close connection between the notions of morphism and congruence, since any homomorphism defines a congruence relation and any congruence relation defines a homomorphism. If, after Bourbaki and MacLane, one considers that the notion of morphism is the central notion of mathematics, one can say that mathematics is in fact a general theory of congruence relations.

The concept of congruence is not very well-known among philosophers. That is why maybe they don't have a clear idea of what is identity. The concept of congruence (in a rather geometrical sense) was used by Suppes to clarify the relation of identity between propositions (see e.g. Suppes, 1986).

One can wonder if it is correct to reduce identity to trivial congruence, i.e. to trivial identity. Are there no other candidates for identity among congruences?

5. Leibniz congruence and the identity of indiscernibles

Another famous one is the top of the lattice, which can be called the *Leibniz congruence*. This congruence obeys the converse of the condition (C):

$$\text{(LC) } a \cong b \Leftarrow R[a,c] \text{ iff } R[b, c].$$

This condition corresponds to Leibniz's principle of identity of indiscernibles and that is why we use the terminology "Leibniz congruence".[4]

There have been a lot of discussions about the *truth* of the principle of identity of indiscernibles. In this mathematical framework

[4] Of course we could have said that trivial congruence is the strongest congruence relation, that it is the top of the lattice, and that Leibniz congruence is the weakest congruence relation, the bottom of the lattice. Barcan chooses this other option. In (Barcan, 1993) her terimonology is very confuse, sometimes tautological and/or nonsensical. For example she writes "Identity is the strongest of the equivalent relations that are transitive, reflexive and symmetrical" (p. 202). This means nothing else than "Identity is the strongest of the equivalent relations that are equivalent relations". She writes also that "Identity is the strongest equivalent relation that a thing bears only to itself" (p. 200). But there is only one relation that a thing bears only to itself. In (Barcus, 1961) she writes: "On the level of individuals, one or perhaps two equivalence relations are customarily present: identity and indiscernability. This does not preclude the introduction of others such as congruence, but the strongest of these is identity" (p. 6). In this terminology it seems that (trivial) identity and indiscernability are not congruence relations, that they are just equivalence relations.

we see that the question should be properly restated as follows: should one consider Leibniz congruence as identity?

Of course Leibniz congruence can be considered as identity, but there is no reasons to consider that this is the only identity. In particular it would be absurd to do so in a structure which is not simple.

A structure is called *simple* iff there is only one congruence relation. In a simple structure therefore the trivial congruence and the Leibniz congruence are the same, they are this only congruence relation. If the structure is not simple the Leibniz congruence is different from the trivial congruence.

In a structure which is not simple, to consider that the principle of identity of indiscernible is true in the sense that Leibniz congruence is the only identity, is to consider that the trivial congruence is not identity, which is quite absurd. In fact in a structure which is not simple, it seems to us that any congruence relation can be considered as an identity, so there is not only one identity, but several identities: trivial identity, Leibniz identity, and other intermediate identities.

One could say that to consider that Leibniz's principle of identity of indiscernibles hold in the physical world is to consider that the physical world is a simple structure.

6. Axiomatization of identity

Let us consider the following first-order formula:

$$(\mathrm{R})\ \forall x\,(x=x)$$

and the the following first-order scheme of formula (φ is any formula):

$$(\mathrm{K})\ \forall x \forall y\,(x=y \to (\varphi x \to \varphi y))\ .$$

A lot of peope say that the formulas (R) and (K) are *axioms* for identity. What do they mean exactly? One must be careful since "axiom" may have different meanings.

One of the meanings is the following. We say that a relation is axiomatizable iff there is a set of formulas such that the models of these formulas are exactly the structures in which this relation holds. If this set of formulas is recursive, we say that the relation is recursively axiomatizable, and if the set is finite, finitely axiomatizable. If the formulas are first-order formulas, we say that the relation is first-order axiomatizable, if they are second-order formulas, second-order axiomatizable, etc.

For example the relation of well-order is not first-order axiomatizable, but is second-order finitely axiomatizable.

(R) and (K) together form a recursive set of first-order formulas. Let us call this set (RK). One can wonder if (RK) is an axiomatization, in the above model-theoretical sense, of identity. To have an answer to this question we must look at the models of (RK).

In a model of (RK), = corresponds to Leibniz congruence and in fact (RK) is equivalent to the following scheme alone:

$$\text{(L) } \forall x \forall y\, (x{=}y \leftrightarrow (\varphi x \leftrightarrow \varphi y))\,.$$

The set (RK) is an axiomatization of Leibniz identity, and therefore not of trivial identity, since in a structure these two identities may be distinct.

Trivial identity itself is not first-order axiomatizable, in the model-theoretical sense. Few textbooks mention this fact, even fewer present a proof of it. One of them is (Hodges, 1983).

Most of the logicians say that they take identity as *primitive*. What does this mean? It means that in the semantical definition of satisfaction they force the symbol "=" to be trivial identity by a definition which is quite tautological: one says that the atomic formula "$x = y$" is satisfied by the sequence (a, b) of elements of a structure iff $a = b$ in the structure, where "=" is the tivial identity.

People rarely mention the fact that they take trivial identity as primitive *because* it is not axiomatizable. After presenting a set of axioms equivalent to (RK), Van Dalen (1980, p. 83) shyly suggests this: "It is important to realize that from the axioms alone, we cannot determine the precise nature of the interpreting relation. We explicitly adopt the convention that "=" will always be interpreted by real equality."

It would be interesting to meditate about the exact nature and meaning of this *convention*. Generally nobody really thinks about it, because in some sense it works: once "=" is taken as primitive, that its semantical interpretation has been forced to be trivial identity, one can axiomatize it in a proof-theoretical sense: if we consider a Hilbert-type proof system for first-order logic and we add the set of formulas (RK), then this proof-system is sound and complete relatively to the model-theoretical semantics in which the interpretation of "=" is trivial identity (this was in fact proven by Gödel himself, cf. Gödel, 1930, theorem VII).

In a future paper we will discuss the question of axiomatization of identity in second-order logic.

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