What is “formal logic”?

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Abstract: In this paper we stress the ambiguity of the expression “formal logic”. We distinguish five meanings of this expression and we explain that logic in its present stage is not necessarily formal in a genuine way.

Keywords: Formal logic, symbolic logic, mathematical logic.

The subject of formal logic when treated by the method of setting up a formalized language is called symbolic logic, or mathematical logic or logistic.
Alonzo Church, 1956, p.56.

0. Introduction

Many people understand the expression “formal logic” as meaning modern mathematical logic by opposition to traditional logic before the revolution that happened in the second part of the 19th century with Boole, Frege and others. But in fact this expression was created by Kant (see Scholz 1931). Some people like to quote an excerpt of the preface of the second edition of the *Critic of pure reason* (1787), where Kant says that formal logic is a finished and closed science: “logic … has not been able to advance a single step, and hence is to all appearances closed and complete”. Retrospectively, this remark by Kant seems pretty ridiculous. One may wonder how such a wise man could have been so wrong. On the other hand it is quite ironic that the expression created by this philosopher has turned to be used to name the new logic that he was not able to prophesy. Of course “formal logic” is not the only expression used to denote the new logic but it is quite popular and widely spread, maybe because it means several things at the same time.

We can distinguish at least five different meanings:

(1) Formal logic in the sense that the validity of inferences depends on the form and not on the matter or meaning.

(2) Formal logic as a formal science by opposition to an empirical science.

(3) Formal logic in the sense of a formalized theory, to be understood in relation with the formalist program promoted by Hilbert, Curry and others.

(4) Formal logic as symbolic logic, a science using symbols rather than ordinary language.

(5) Formal logic as mathematical logic, logic developed by the use of mathematical concepts or/and the logic of mathematics.

We will explain in more details these 5 senses and we will see that classical logic (propositional logic/first-order logic) is formal in these 5 senses. However we will also show that these 5 senses are quite independent from each other and that logic should not necessarily be formal in these 5 senses. According to the present new orientation of logic, it seems that logic is formal only in the sense of (4) and (5), therefore formal in no genuine way, as we will explain.

> *The discovery of the formal treatment of logic, i.e. of the possibility of describing deductive reasoning with sentences in terms of their form, appears with Aristotle.*
>
> Stephen Kleene, 1952, p.61

1. Logical form

It seems that the idea that the validity of an inference depends on its form and not on its matter or meaning is due to Aristotle. This is one striking feature of his logic. From this point of view Aristotle can be considered as the grand father of formal logic. It seems also that the logical revolution of the 19th century didn't challenge this point, and that on the contrary it reinforced it through the rise of symbolism, formalism and mathematization of logic. From this point of view there is a very strong connection between ancient and modern logic (in its standard trend): modern logic, like ancient logic, is still based on logical form, but the way to characterize it has changed.

The characterization of the notion of logical form in modern logic was not easy. This notion first appeared through the so-called rule of substitution whose status was quite confusing (this rule was first stated, incorrectly, by Couturat 1905, a sophisticated definition of this notion was formulated by Pogorzelski and Prucnal 1975). In this context, a well-known theorem of classical propositional logic says that if a formula is a theorem, any substitution of it is also a theorem. A substitution consists of replacing uniformously in a formula, an atomic formula by an arbitrary formula. The notion of substitution leads in fact to the notion of scheme of formula (due to von Neumann 1927, see Church, 1956, p.158). Once we have this concept, we can present a proof system where axioms and rules are schemes, then the substitution theorem appears rather as an axiom, expressing the formal character of logic.

In fact in the 1950s the substitution theorem was explicitly stated as an axiom in the abstract definition of logic by Los and Suszko (1958). They defined a logic as structural if it obeys this axiom and they showed that all the known logics, classical or non classical are structural. Curiously even logics, like relevant logics, whose aim is to take in account the meaning in inference processes, are structural. This seems quite absurd. One may think that a logic of meaning must be non structural. But of course this is not a sufficient condition. It is easy to find some non structural logics which have nothing to do with meaning. A simple example is anti-classical logic, i.e. the set of formulas and inferences which do not hold in classical logic. An atomic formula does not hold in classical logic, but a substitution of it can hold.

Nevertheless the attempt of relevant logicians shows that there is a strong insatisfaction with the Aristotle's paradigm of logical form. Wittgenstein himself claimed he realized that all logic was wrong, the day he saw that it cannot explain a simple inference as "If it rains, the road is wet". Nowadays in Artificial Intelligence, people are trying to describe and characterize non formal inferences, through Semantic Networks or other tools. This kind of logic is not formal anymore in the Aristotelian sense. Finally let us note that the precise study of the law of substitution has led Haskell Curry (1929) to a complete reformulation of logic: combinatory logic.

> *The grand aim of all science is to cover the greatest number of empirical facts by logical deduction from the smallest number of hypotheses or axioms.*
>
> Albert Einstein

2. Formal vs. Empirical

The distinction between formal sciences and empirical sciences may appear simple and obvious, but in fact it is based on a philosophical theory mainly due to Kant related to his famous analytic/synthetic distinction, which nowadays may sound quite obsolete.

According to this distinction, formal sciences are mathematics and logic[1] by opposition to empirical sciences (all the other sciences): physics, biology, sociology, etc. The idea is that empirical sciences have to do with experience, contact with the "external world". One fundamental idea of Kant is that mathematics is not an empirical science, because it is based on pure intuitions of space and time, which are not part of the world but shape the world. Logic also has to be considered in this way: as a world shaping device. According to Kant the laws of logic are independent of experiences and are revealed quite directly. This may be one of the reasons why he thought that it was so natural they had been described once and for all by Aristotle.

Now we have a different perspective: we know that it is not so simple to discover the "laws" of reasoning. The study of reasoning needs some observations and some experiences in the same way as the study of language, memory and so on. During many years the neo-Kantian paradigm of antipsychologism promoted in particular by Frege has dominated the research in logic. Maybe for this reason, works developed by people like Piaget, were not taken seriously in account by logicians. But it seems that nowadays, through the development of Artificial Intelligence and Cognitive Science, the paradigm of formal logic as a non empirical science is coming to an end.

The standard conception of modern logic is not necessarily opposed to a Kantian vision: one may think that classical is just a more precise account of the laws of thought and that it just uses mathematics, which is also a formal science. But these last years different models of reasoning have been presented which are quite different from classical logic and which contrary to it, involve experimentation (see e.g. Suppes and Beziau 2004).

This does not necessarily mean that logic is an empirical science in the same sense as physics or biology, but this new tendency seriously challenges the distinction between formal and empirical science.

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[1]i.e. formal logic, Kant makes the distinction between formal logic (he coined the expression as we have seen) and transcendental logic, which is part of philosophy.

A word or an image is symbolic when it implies more than its obvious and immediate meaning,

Carl G.Jung 1964, p.4.

3. Is symbolic logic formal?

The expression “symbolic logic” was quite popular at some point, it was in particular used by people like John Venn (1881), Charles Dodgson (Lewis Carroll) (1897), Lewis (1918), Lewis and Langford (1932), etc. It was definitively crystallized through the “Association of Symbolic Logic” and the correlated “Journal of Symbolic Logic” (1936).

The expression “symbolic logic” is highly ambiguous. As it is known, mathematics uses symbols, it is rather written in a symbolic language than natural language. And when people, like Boole, started to use mathematics to deal with logic, they also started to use some symbols, at first symbols from mathematics and then symbols were created especially for logic, like symbols for negation, conjunction, disjunction, quantifiers, Frege’s stroke, etc. (logic symbolism has been collected by Feys and Fitch 1969).

But mathematics and logic are not the only fields of human activities where symbols are used. Symbols are also used in religion, art, astrology, etc. At first it seems that there are no relations between religious symbols and mathematical symbols, in fact many people think that these two kinds of symbols are completely opposed and that they bear the same name only by accident: mathematical symbols would be connected with reason, precision and objectivity, by opposition to religious symbols appealing to emotion, ambiguity and subjectivity.

However there is a common feature to both mathematical and religious symbols. This is not necessarily easy to understand for someone who has been brainwashed by the formalist ideology according to which “symbols are considered wholly objectively ... are themselves the ultimate objects, and are not being used to refer to something other than themselves; thus they are objects without interpretation or meaning” (Kleene, 1952). Let us emphasize that this use of “symbol” is artificial and uncommon. From the philology and semiotics point of view, a symbol, is a sign in which there is a connection between the signifier and the signified. An alphabetical letter is not a symbol, a Chinese ideogram is. The symbolic degree of a sign may vary, just think about traffic signs: some are highly symbolic, other are less, or not at all. One may wonder to which point mathematical signs are symbolic. Roman signs for numbers are surely more symbolic than Arabic signs, but even these last ones are quite symbolic in the sense that they clearly show the basis 10 construction.

Symbols in mathematics are used to abbreviate, but to abbreviate in a meaningful way. People who don’t understand this, don’t know how to write mathematics (cf Halmos, 1970). Symbols in mathematics do not reduce to simple signs as the ones used for numbers, they include visual representations of functions, commutative diagrams (category theory), fractals, etc.

Frege’s ideography (1879) is obviously symbolic in this sense. The wave of formalism nearly led the people to forget about this, replacing Frege’s system by some totally blind and meaningless formal systems, which can be called symbolic only by derision. But soon, symbolism in its true sense was reintroduced in proof theory:

Gentzen’s systems, semantic tableaux, trees, etc. And recently people have been constructing some proof systems based on Venn’s diagrams, showing that they are not only heuristic devices but can be perfectly used for developing reasonings which are at the same time rigorous and meaningful (Barwise and Hammer 1994). This is a happy end for the expression “symbolic logic” which seems to have been coined by Venn.

Formal logic, in the Aristotelian sense – logic form, was not really symbolic, but Aristotle used some schematic letters to represent some undetermined concepts, representing thereof in an “anonymous” way the matter or content of reasoning. Later, syllogisms have been represented in a more symbolic way using diagrams, by Euler and other people.

Symbolic logic, as a logic using symbols, doesn’t necessarily have to be formal in the sense of Aristotle. Semantic networks are symbolic devices and they are used to promote a view of logic which does not fit in the Aristotelian paradigm of formal logic (see e.g. Lehman 1992).

To Hilbert is due the emphasis that strict formalization of a theory involves the total abstraction from meaning.
Stephen Kleene, 1952, p.62

4. Is logic formalized or formalizable?

When one speaks about the formalization of a theory, it generally involves two steps. Step-1: to develop a formal language. Step-2: to give a set of axioms and rules written in this language, such that all the truths of the theory, but only them, can be derived mechanically from the axioms using the rules. We take “theory” here in a broad sense - including things like the theory of relativity, the theory of models, arithmetic - not in its restricted logical sense meaning a set of sentences, or a deductively closed set of sentences.

Formalization was applied mainly to logic and mathematics one hundred years ago. The formalist program, especially promoted by Hilbert, was the attempt to formalize the whole mathematics. The fall of Hilbert’s house, to use Girard’s expression (Girard 1986), is due to Gödel, Tarski and Church’s results about the incompleteness, indefinability of truth and undecidability of simple mathematical theories like arithmetic. One may think that the problem is mainly with Step-2. But in fact there are already some problems with Step-1, because the construction of a very simple formal language, like the language of propositional logic is not really possible according to the norms of the formalist program. It is not possible to strictly formalize this language, because we cannot get a complete axiomatization of it in first-order logic (see Béziau 1999). Formalists erroneously thought that to set up a formal language only very simple operations were required.

Other striking results are about the axiomatization of simple notions such as identity: it has been shown that the identity relation (the “diagonal”) is not axiomatizable in first-order logic in the same sense that e.g. the notion of well-ordering is not first-order axiomatizable (see e.g. Hodges 1983). These facts despite their deep meaning are mainly ignored, especially by philosophers of logic.

But the truth is that logic itself is not formalizable. We can say that logic is not formal in the sense that it is not formalizable, it is not a formal system according to the precise definition of this notion related to the theory of recursion: “due to A.M.Turing’s work (Turing 1937) a precise and unquestionably adequate definition of the general notion of formal system can now be given. In my opinion, the term “formal system” or “formalism” should never be used for anything but this notion. ... characteristic property [of formal systems] is that reasoning in them, in principle, can be completely replaced by mechanical devices” (Gödel, Note added in 1963 to Gödel, 1931, in Heijenoort, 1966, p.616).

Leibniz had the idea of a mechanical system which would substitute the thought process. He can in this sense be considered as a forerunner of formalism and formal logic. Formalists thought that it was possible to achieve their goal by rejecting meaning. But the goal was not reached. However, it is true that the manipulation of signs, not taking in account the meaning can turn computation easier, as anybody who has performed the basic algorithms of arithmetic knows. But if logic, the reasoning process, does not reduce to computation, it is not clear at all that the formal approach is efficient unless it is to build truth-tables, or to perform algorithms of disjunctive normal form, etc. But to believe that logic reduces to such games would be the same as to believe that mathematics reduces to algorithm for addition and multiplication.

If we think that mathematical reasoning doesn’t reduce to a formal process, then it seems unlikely that logic, as the study of such reasoning, would be a formal process. And even if we deal with logic as the study of computable reasoning, such theory is not a formalized or formalizable theory.

Finally it is important to emphasize that the notion of axiomatization is not necessarily related with formalization, as the Step-1/Step-2 formalist recipe may suggest. Axioms were presented well before the development of any formalized language. Moreover, we can think “axiomatically” in a semantic way, in second order logic, etc. (cf Dedekind’s second-order axiomatization of arithmetic in an “informal” language, 1888).

A knowledge of the theory and practice of formal language might be a help for writing with precision, especially to students whose talents are not mathematical, but it is of no help at all for writing with clarity.

Paul Halmos, 1985, p.164.

5. Is mathematical logic formal?

Modern logic is mathematical, in the sense that it uses mathematical tools and concepts, there is no doubt about this, but should it be called for this reason “formal”? Not necessarily. If we think that mathematics is not genuinely formal, there is no reason to say that mathematical logic is.

Mathematics can be said to be formal for at least four reasons: the fact that it is a formal science by opposition to an empirical science, that it is reducible to logic and logical form, that it uses symbolism, that it is abstract. We have already discussed many points related to the first three reasons in other sections. If we don’t support a logico-formalist philosophy of mathematics, the first three reasons have to be rejected.

One central feature of mathematics seems abstraction, but why should abstraction be related to any “formal ontology”? This relation between the abstract and the formal has probably to be traced back to a certain interpretation or, better, deviation of Plato’s philosophy. At some point, Plato’s “eidos” has been translated by “form”, perhaps because form meaning shape is an easy representation of what could be an “eidos”. But it is clear that concepts, whether they exist or not by themselves, have especially nothing to do with a notion of form. Mathematical concepts are no exception, except geometrical concepts which are related to form in a spatial sense. Why should the number 3 be considered as a common “form” of sets having 3 elements?

However, it does not seem so easy to throw out the concept of form, when talking about mathematics. Form is like a multi-headed dragon: cut one head, three more heads grow. One may reject the formalist approach to mathematics and prefer the structuralist viewpoint (only a mythological character like Bourbaki was able to reconcile these two opposite approaches). But the notion of form reappears again, because the basic notion of mathematics as a theory of structures is the notion of “morphism”, a Greek word meaning “form”. Two structures are isomorphic if they have the same form. Of course, the notion of form here is quite different from the one which appears in the formalist ideology, but maybe not so different from the one which appears in the idea of logical form. In fact, this is no coincidence if the notion of logical form was characterized by Los and Suszko (1958) using the notion of endomorphism.

The notion of mathematical structure and his inseparable sister, the notion of morphism, can be considered as the basic tools for the development of mathematical logic (Porte 1965) even for logics not formal in the Aristotelian sense (Béziau 1994). But this would be quite ambiguous to say that logic is formal for this reason. Due to the Promethean character of the notion of form, it seems better not to use it to characterize mathematics (as did MacLane 1986), it would be better to speak about structuralist mathematics and structuralist mathematical logic.

6. Conclusion

Nowadays the expression “formal logic” looks quite old-fashion and out-of-date. It may sound quite charming to the ears of an old English lady reading De Morgan while drinking her tea. But due to the variety of possible meanings of this expression, leading to confusing ambiguity and due to the fact that logic recently is not formal according to many of these meanings, it seems better not to use it anymore.

One may speak about “mathematical logic”, but this expression may also be ambiguous (it can mean the logic of mathematics or logic using mathematical concepts). The best way seems simply to speak about logic *tout court*. In the same way that we speak about physics *tout court*, and not mathematical physics, even if nowadays mathematics is very much used in physics.

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