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Journal of Applied Logic 3 (2005) 7–14

JOURNAL OF APPLIED LOGIC

www.elsevier.com/locate/jal

Paraconsistent logic from a modal viewpoint ☆

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Available online 25 August 2004

Abstract

In this paper we study paraconsistent negation as a modal operator, considering the fact that the classical negation of necessity has a paraconsistent behavior. We examine this operator on the one hand in the modal logic $S5$ and on the other hand in some new four-valued modal logics.

Keywords: Modal logic; Paraconsistent logic; Negation; Square of opposition

1. Introduction

In this paper we show how the notion of paraconsistent negation can be thought from a modal viewpoint.[1]

In the next section we have a look at the squares of oppositions and modalities and we point out that one of the corners of the square has no name in natural language. In fact this nameless corner is a paraconsistent negation.

The square of modalities is a general view on modalities independent of a particular logic. In the next two sections we study the nameless modality, which has the feature of a paraconsistent negation, in the context of two definite modal logics. First in the context of

☆ Work supported by a grant of the Swiss Science Foundation and the author is a member of the LOCIA project (CNPq/Brazil).

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[1] More precisely, from a *new* modal viewpoint. Jaśkowski's approach (cf. [11]) is also connected with modal logic. However our starting point here is quite different, although there are some connections at the semantical level, see Section 3.3.

1570-8683/$ – see front matter
doi:10.1016/j.jal.2004.07.009

the famous modal logic S5, second in the context of a new four-valued modal logic of our own.

The modal interpretation of paraconsistent negation is very interesting from the point of view of the intuitive understanding of paraconsistency and is a good basis for application of paraconsistent logic to natural language, linguistics and computation.

The aim of this paper is to offer an hint on the modal approach to paraconsistency. We present the basic idea, detailed work will be carried on elsewhere [3–5]. We also hope that it will act as a stimulus for other researchers and open a new area of investigation in paraconsistent logic.

2. The nameless corner of the squares of oppositions and modalities

The square of oppositions is a famous concept of traditional logic, coming directly from Aristotle's logic (although the square itself does not appear in Aristotle, see [12]). There are several variations of it, not necessarily equivalent.

According to modern first-order logic, the four corners of the square correspond to $\forall$, $\exists$, $\neg\forall$ and $\neg\exists$. The first two quantifiers are read "all" and "some" (or "there exists"), and in English the word corresponding to the quantifier $\neg\exists$ is "none". However in English no primitive word corresponds to $\neg\forall$. Recent researches show that there is no natural language in which there is a primitive word for this quantifier (cf. [10]). This has led some people to reject the square of oppositions, arguing that the nameless corner of the square is meaningless. We will show here that we don't need to reject more than two thousands years of logical tradition and that we can find a meaningful interpretation of the nameless corner.

We will find a solution looking at the modal version of the square of oppositions, the so-called square of modalities, which can be found, for example, in a paper of Łukasiewicz (cf. [13]). The four corners of the modal square are $\Box$, $\Diamond$, $\neg\Box$, $\neg\Diamond$.

The square of modalities coincides in some sense with the square of opposition, at least one precise connection can be made via Wajsberg's theorem [16]. And furthermore, we have here a situation similar to the case of the quantifiers, since we have words in natural language for three of the modalities, "necessary", "possible" and "impossible", but there is no word for the modality $\neg\Box$ which corresponds to the nameless quantifier $\neg\forall$.[2]

As it has been shown by Gödel, the modality $\neg\Diamond$ in $S4$, the impossible, corresponds to intuitionistic negation (cf. [9]). If we think of this modality independently of a specific modal logic, it is a *paracomplete negation*, i.e. a negation for which the law of excluded middle does not hold.[3]

Duals of paracomplete negations are paraconsistent negations. Due to the relation of $\neg\Box$ and $\neg\Diamond$ in the square, we can guess that these two modalities are dual and that $\neg\Box$ is in general a paraconsistent negation.

[2] One may think erroneously that this nameless modality is "contingency", contingency is in fact defined as $\Diamond a \wedge \neg\Box a$.

[3] K. Došen has written several papers in this direction, i.e. the study of negation as impossibility, see, e.g. [7].

3. The nameless corner from the point of view of $S5$

3.1. *Basic properties of the modality $\neg\Box$ in $S5$*

The operator $\sim$ defined by $\neg\Box$ has the following properties in $S5$: we have formulas a and b such that,

$$a, {\sim}a \nvdash b$$

$$a, {\sim}a \nvdash \neg b$$

This shows that $\neg\Box$ obeys the basic negative requirements in order to be considered as a paraconsistent negation. Let us see now its positive properties.[4]

The following are theorems:

$$a \vee {\sim}a$$

$$\sim(a \wedge {\sim}a)$$

$$(a \to {\sim}a) \to {\sim}a$$

$$({\sim}a \to a) \to a$$

$$\sim(a \wedge b) \leftrightarrow ({\sim}a \vee {\sim}b)$$

$$\sim({\sim}a \wedge {\sim}b) \leftrightarrow (a \vee b)$$

$$\sim(a \wedge {\sim}b) \leftrightarrow ({\sim}a \vee b)$$

$$\sim({\sim}a \wedge b) \leftrightarrow (a \vee {\sim}b)$$

And we have the following theorems but not their converses:

$$(a \to b) \to ({\sim}a \vee b)$$

$${\sim}{\sim}a \to a$$

$$\sim({\sim}a \vee b) \to (a \wedge {\sim}b)$$

$$\sim(a \vee b) \to ({\sim}a \wedge {\sim}b)$$

$$\sim(a \vee {\sim}b) \to ({\sim}a \wedge b)$$

$$\sim({\sim}a \vee {\sim}b) \to (a \wedge b)$$

Another important feature is that the bi-implication is a congruence relation in $S5$, in particular we have:

$$\text{if } \vdash a \leftrightarrow b \text{ then } \vdash {\sim}a \leftrightarrow {\sim}b$$

Finally, an interesting fact is that we can reconstruct $S5$ taking as primitive connectives: $\wedge$, $\to$ and $\sim$, i.e. without classical negation or a standard modality.

[4] For a discussion about positive and negative requirements for a paraconsistent negation, see [2].

3.2. *The semantics of* $\neg\Box$ *in* $S5$

It is easy to define a semantics for this negation. If we consider the possible worlds semantics for $S5$ with a universal relation of accessibility, we have in a given frame $\mathcal{K}$:

$\neg\Box a$ is false in the world W,

iff $\Box a$ is true in the world W,

iff a is true in every world of $\mathcal{K}$.

Therefore $\sim a$ can be semantically defined in $S5$ by the following condition:

$\sim a$ is false in the world W iff a is true in every world of $\mathcal{K}$.

If one wants to study $\neg\Box$ in a given Kripke semantics other than the one for $S5$, one can take the following condition in a given frame $\mathcal{K}$ with an accessibility relation R:

$\sim a$ is false in the world W iff a is true in every world of $\mathcal{K}$ accessible from W.

In $S4$ this is the dual of the condition which defines the intuitionistic negation $\approx$, which is the following:

$\approx a$ is true in the world W iff a is false in every world of $\mathcal{K}$ accessible from W.

3.3. *Interpretation of* $\neg\Box$ *in* $S5$

Combining the possible worlds of $S5$ with an idea connected to Jaśkowski's discussive logic (cf. [11]) we have a quite intuitive interpretation of the paraconsistent negation $\neg\Box$:

We can imagine that a frame is a discussion group and that the worlds are members or agents of the discussion group. The paraconsistent negation of a is false for an agent of the discussion group if and only if every agent of the group agrees that a is true. This means in particular that if every agent of the group agrees that a is true, then the paraconsistent negation of a is false for the group, i.e. for every agent of the group.

The truth or falsity of this paraconsistent negation is holistic, it depends of the opinion of the other agents. Contrary to the case of Jaśkowski's logic, an agent can think that a and its paraconsistent negation $\sim a$ are both false. (Compare with [6].)

Another interesting feature of $\neg\Box$ is related to double negation. In natural language double negation is often used to emphasize a sentence in such a way as if it was stronger than simple affirmation, as in the following example:

It is not true that God does not exist.

In a logic like $S5$ in which we have

$$\sim\sim a \rightarrow a$$

but not

$$a \rightarrow \sim\sim a$$

double (paraconsistent) negation is really stronger than simple affirmation. The reason why this is the case in $S5$, is that double (paraconsistent) negation means necessity, as we have:

$$\Diamond\Box a \leftrightarrow \Box a$$

and considering that:

$$\Diamond\neg\Diamond\neg a \leftrightarrow \Diamond\Box a$$

$$\neg\Box\neg\Box a \leftrightarrow \Diamond\Box a$$

we have:

$$\sim\sim a \leftrightarrow \Box a.$$

Therefore the above double negated sentence means from the point of view of the paraconsistent negation of $S5$:

God necessarily exists.

4. The nameless corner from the point of view of the four-valued modal logic $M4$

Several four-valued semantics have been presented to define a negation which is paraconsistent, the most famous being due to Belnap [1]. Our starting point here is different, because first we construct a four-valued semantics for a modal logic taking $\Box$ and $\Diamond$ as primitive, and then we define the semantics of a paraconsistent negation following the general idea of the square by defining this negation as $\neg\Box$.

4.1. The four-valued modal logic $M4$

Łukasiewicz has proposed a four-valued semantics for a logic of necessity and possibility (cf. [13], see also [8]), however this semantics generates a logic with quite strange properties, strange at least from the viewpoint of a standard modal logic like $S5$.

Our idea here is to construct a modal logic with a four-valued semantics having more standard properties than Łukasiewicz's logic.

We consider a set of four-values, two non-designated values, 0^- and 0^+, and two designated values, 1^- and 1^+. These values are ordered by the following linear order: $0^- \prec 0^+ \prec 1^- \prec 1^+$.

Classical negation, possibility and necessity are defined by Table 1.

Table 1

a	$\neg a$	$\Box a$	$\Diamond a$
0^-	1^+	0^-	0^-
0^+	1^-	0^-	1^+
1^-	0^+	0^-	1^+
1^+	0^-	1^+	1^+

Table 2

p	$\neg p$	$p \vee \neg p$	$\Box(p \vee \neg p)$
0^-	1^+	1^+	1^+
0^+	1^-	1^-	0^-
1^-	0^+	1^-	0^-
1^+	0^-	1^+	1^+

Table 3

a	$\neg a$	$\Box a$	$\neg\Box a$
0^-	1^+	0^-	1^+
0^+	1^-	0^-	1^+
1^-	0^+	0^-	1^+
1^+	0^-	1^+	0^-

Table 4

p	$\neg\Box p$	$\Box\neg\Box p$	$\neg\Box\neg\Box p$
0^-	1^+	1^+	0^-
0^+	1^+	1^+	0^-
1^-	1^+	1^+	0^-
1^+	0^-	0^-	1^+

Conjunction and disjunction are defined in the usual way by the operators *min* and *max*. Implication is defined as $\neg a \vee b$.

In the modal logic $M4$, necessity distributes over conjunction, and possibility over disjunction, Kripke's law is valid. We have also reduction of modalities and the replacement theorem is valid. In fact $M4$ has almost all properties of $S5$, minus the rule of necessity.

A typical example of the failure of the rule of necessity is the following: $p \vee \neg p$ is a tautology of $M4$ but $\Box(p \vee \neg p)$ is not. This is shown by Table 2.

The fact that the rule of necessity is not valid for $M4$ can be seen as a serious defect, however Łukasiewicz has argued at length against the validity of such a rule (see [14]). In $M4$ we also have a feature similar to Łukasiewicz's logic: necessity distributes over disjunction and possibility over disjunction.

4.2. *Properties of $\neg\Box$ in $M4$*

The semantics of $\neg\Box$ in $M4$ is explained and defined by Table 3.

This negation has the same feature as the negation of Sette's logic $P1$, it is paraconsistent only at the atomic level since it collapses the four values in two values which have classical behavior. This is shown by Table 4.

Therefore, if we use the abbreviation $\sim$ for $\neg\Box$, we have

$$p, \sim p \nvdash q$$

for a given atomic formula q, but

$$\sim p, \sim\sim p \vdash a$$

for any formula a.

4.3. *Interpretation of $\neg\Box$ in $M4$ and another proposal*

The four-values can be interpreted as follows:

0^- means necessarily false,
0^+ means possibly false,
1^- means possibly true,
1^+ means necessarily true.[5]

Following this interpretation, we can explain the behavior of the paraconsistent negation $\sim$ defined by $\neg\Box$ as follows:

When a is false, $\sim a$ is "necessarily true", since we don't want here a paracomplete negation and since, due to the definition of necessity, $\sim a$ cannot be "possibly true". And when a is "necessarily true", then $\sim a$ is "necessarily false". The controversial point is when a is "possibly true", then one could expect $\sim a$ to be "possibly false" or "possibly true", but in fact these two intermediate values are eliminated by the definition of necessity. Now if we choose "necessarily false", we will have a paracomplete negation, therefore "necessarily true" is the only possibility.

This interpretation seems quite strange, the negation defined by Table 5 will have a more intuitive interpretation.

This paraconsistent negation can be constructed from classical negation and necessity defined in Table 6.

The modal logic defined by this table is somewhat weaker than $S5$, in particular we don't have reduction of modalities. However it is relatively intuitive and obeys the conditions given by the square of modalities.

Finally the reader may point out that the paraconsistent negations defined with our set of four values, could have been defined with only three values. This is not totally false, but the full meaning of these paraconsistent negations is based on the whole framework which includes also paracomplete negations and negations that are both paracomplete and paraconsistent. From this point of view our four valued logic is much richer than Belnap's. $S5$ also is a very rich logic where it is possible to define not only a paraconsistent negation, but also a paracomplete one dual of it and a negation that is both paraconsistent and paracomplete.

Our conclusion is therefore that paraconsistent negations constructed in a modal perspective, following the square of modalities, have a nice architecture. Long live Aristotle.

Table 5

a	$\sim a$
0^-	1^+
0^+	1^-
1^-	1^-
1^+	0^-

Table 6

a	$\neg a$	$\Box a$	$\neg\Box a$
0^-	1^+	0^-	1^+
0^+	1^-	0^+	1^-
1^-	0^+	0^+	1^-
1^+	0^-	1^+	0^-

[5] Compare with [15, p. 98], where "contingently" is used instead of "possibly".

Acknowledgements

Topics presented in this paper were improved by discussions with many people in particular Johan Van Benthem, Witold Bychovski, Newton da Costa, João Marcos and Patrick Suppes. Five anonymous referees made also valuable comments.

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