A NEW FOUR-VALUED APPROACH TO MODAL LOGIC

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Abstract
In this paper several systems of modal logic based on four-valued matrices are presented. We start with pure modal logics, i.e. modal logics with modal operators as the only operators, using the Polish framework of structural consequence relation. We show that with a four-valued matrix we can define modal operators which have the same behavior as in pure S5 (S5 with only modal operators). We then present modal logics with conjunction and disjunction based on four-valued matrices. We show that if we use partial order instead of linear order, we are avoiding Łukasiewicz's paradox. We then introduce classical negation and we show than defining implication in the usual way using negation and disjunction Kripke law is valid using either linear or partial order. On the other hand we show that the difference between linear and partial order appears at the level of the excluded middle and the replacement theorem.

1. *Introduction*

In this paper we present several systems of modal logic based on four-valued semantics. Łukasiewicz introduced many-valued logic to deal with possibility. In his first three-valued logic, he calls the third value "possible". Many-valuedness was in fact the first formal semantic treatment of modalities. However after the negative result of Dugundji [5] showing that standard modal logics cannot be characterized by finite matrices and the incredible success of Kripke semantics, this approach has been marginalized. In the last trendy books on modal logic, it is not even taken in consideration.

Maybe Łukasiewicz himself has to be considered as responsible for this disaster. He persisted to defend his modal system based on three-valuedness with some unexpected properties and at the end of his life he presented a

quite awkward four-valued system of modal logic [9]. Perhaps Łukasiewicz's four-valued logic does not really make sense.

In this paper we try to rethink from the start the four-valued approach to modal logic. Our idea is to consider basic properties for modalities expressed in the Polish framework and to see what kind of matrices are suitable for them. By Polish framework we mean, following Łoś and Suszko [7], structural consequence relations, i.e. consequence relations, defined on absolutely free algebras, obeying the three axioms of Tarski (identity, monotony and transitivity) plus the axiom of substitution.

We follow a step by step scheme, starting from very simple properties and going progressively to more complex ones. In a first section, we explain that two and three values matrices do not work to characterize basic properties of modal logic but we show that four values matrices do the job and we explain how we can interpret these four values.

2. *Basic modal logics*

2.1. *Definition and basic properties*

DEFINITION A *basic modal logic* is a logic with two operators $\square$ (necessity) and $\diamond$ (possibility) verifying the following axioms:

(11) $\square a \vdash a$ (21) $a \vdash \diamond a$

(12) $a \nvdash \square a$ (22) $\diamond a \nvdash a$

THEOREM 1 *In any basic modal logic we have:*

(31) $\square a \vdash \diamond a$ (32) $\diamond a \nvdash \square a$

Proof. (31) results from (11), (21) and transitivity. (32) results from (11), (22) and transitivity.

2.2. *2-valued and 3-valued matrices are not enough for basic modal logics*

We will see in this section that there are no reasonable matrices whose cardinality is strictly inferior to 4 that can be used to deal with basic modal logics.

In the case of two-valued matrices, it is easy to see that the only solution to have a basic modal logic is given by the following table:

$\Box a$	a	$\Diamond a$
0	0	1
0	1	1

TABLE 1

We are therefore forced to treat $\Box a$ as something always false, whatever a is, and $\Diamond a$ as something always true, whatever a is. This means that nothing is necessary and everything is possible. It seems difficult to sustain such trivial conception of modalities. But if everything is possible, this option is possible...

Now let us have a look at what happens going to three-valued matrices. The third value can be designated or not. Let us suppose that it is non-designated. We will denote the two non-designated values by 0^- or 0^+, and denote by 0 and call false any non-designated values. We will denote the only designated value by 1 and call it true.

The following table describes the only choices we have for the values of $\Box a$ and some choices we can have for $\Diamond a$ (we are using plural for choice since the table below represents a set of tables we get using the variety of falsity).

$\Box a$	a	$\Diamond a$
0	0^-	0
0	0^+	1
0	1	1

TABLE 2

Although we may in this case define $\Diamond a$ in a reasonable way, we are forced here again to have a trivial conception of necessity, i.e. $\Box a$ is always false. If we had taken two designated values and one non-designated one, the situation would have been dual: a reasonable definition of necessity but a trivial definition of $\Diamond a$. Therefore we have to multiply the values...

Łukasiewicz at first had a system with possibility defined as in TABLE 2. He was calling the value 0^+ itself possibility. Maybe he was not aware that this system was not really working because he was not considering necessity. In his three-valued system the connective of possibility can be defined with the help of other connectives, so it generally does not appear, and possibility only appears as a truth-value. Only later on, in his four-valued system, the connective of possibility systematically appears. And in this system the notion of necessity also is considered as a connective.

2.3. *4-valued matrices are enough for basic modal logics*

If we consider four-values with only one designated value or only one non-designated value, we will face the same problem as with three values. If we have only one designated value, we have a trivial definition of necessity and if we have only one non-designated value we have a trivial definition of possibility.

We consider a set of four-values, two non-designated values, 0^- and 0^+, and two designated values, 1^- and 1^+. Let us denote by 0 and call false any non-designated value, i.e. 0^- or 0^+, and let us denote by 1 and call true any designated value, i.e. 1^- or 1^+.

THEOREM 2 *The logics defined by the tables obeying the following conditions are basic modal logics.*

$\square a$	a	$\diamond a$
0	0^-	0
0	0^+	1
0	1^-	1
1	1^+	1

TABLE 3

Proof. Left to the reader.

THEOREM 3 *There are no other possibilities.*

Proof. If we want to avoid a trivial definition of necessity and/or a trivial definition of possibility, the only alternative would be to do some switching between 1^- and 1^+ and/or between 0^- and 0^+. Since these values in this context have no specificities, they are not serious alternatives.

We call M4 a modal logic defined by TABLE 3.

2.4. *Interpretation of the 4 values*

The four values can be interpreted as follows:

0^-	Necessarily false
0^+	Possibly false
1^-	Possibly true
1^+	Necessarily true

When a is necessarily false, it is not possible for a to be true, which can be interpreted as $\diamond a$ is false, but when a is only possibly false, we can consider that $\diamond a$ is true. This justifies the following part of TABLE 3:

a	$\diamond a$
0^-	0
0^+	1

Dually, when a is necessarily true, it is not possible for a to be false, which can be interpreted as $\Box a$ is true, but when a is only possibly true, we can consider that $\Box a$ is false. This justifies the following part of TABLE 3:

$\Box a$	a
0	1^-
1	1^+

Necessary true can be interpreted as true in all circumstances, or in all possible worlds, or formally true (these three alternatives are not necessarily equivalent), by opposition to possibly true that can be interpreted as true in some circumstances, or in some possible worlds, or factually true. As an example, we can say that it is possibly true (because factually true) that snow is cold and that it is necessary true (because formally true according to classical logic) that snow is blue or not blue.

3. *Reduction of modalities*

3.1. *Reduction of repetition of the same modality*

Since in a basic modal logic, due to axioms (11) and (21), we already have
$$\Box\Box a \vdash \Box a \qquad\qquad \diamond a \vdash \diamond\diamond a$$
to get reduction of repetition in order to have
$$\Box\Box a \dashv\vdash \Box a \qquad\qquad \diamond a \dashv\vdash \diamond\diamond a$$
we only need to have the two following additional axioms:
$$\Box a \vdash \Box\Box a \qquad\qquad \diamond\diamond a \vdash \diamond a$$
It is easy to check that the following table gives sufficient and necessary conditions to have such axioms.

$\square a$	a	$\diamond a$
0	0^-	0^-
0	0^+	1
0	1^-	1
1^+	1^+	1

TABLE 4

3.2. *Reduction of composition of possibility and necessity*

Now let us have a look at compositions of possibility and necessity, i.e. $\square\diamond$ and $\diamond\square$

Due to axioms (11) and (21), we already have

$\square\diamond a \vdash \diamond a$ $\qquad\qquad$ $\square a \vdash \diamond\square a$

One may want to have the converse of these two axioms

$\diamond a \vdash \square\diamond a$ $\qquad\qquad$ $\diamond\square a \vdash \square a$

so that we have

$\square\diamond a \dashv\vdash \diamond a$ $\qquad\qquad$ $\square a \dashv\vdash \diamond\square a$

It is easy to check that the following table gives sufficient and necessary conditions to have this situation.

$\square a$	a	$\diamond a$
0^-	0^-	0
0^-	0^+	1^+
0^-	1^-	1^+
1	1^+	1^+

TABLE 5

3.3. *Reduction of repetition and composition*

We can put TABLE 4 and TABLE 5 together, we then get the following table:

$\square a$	a	$\diamond a$
0^-	0^-	0^-
0^-	0^+	1^+
0^-	1^-	1^+
1^+	1^+	1^+

TABLE 6

Note that in this logic, compound modalities can have only two truth-values, 0^- or 1^+. In this logic we have the same disposition of modalities as

in S5:

$$\begin{array}{ccc} & \Box\phi & \\ & & \\ \Box\Diamond\phi \longleftarrow & a \longleftarrow & \Diamond\Box\phi \\ & & \\ & \Diamond\phi & \end{array}$$
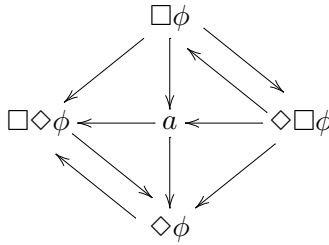
4. *Codi modal logics*

4.1. *Axioms for codi modal logics*

We deal now with modal logics in which we have a classical conjunction and a classical disjunction.

DEFINITION A *codi modal logic* is a modal logic verifying the following conditions.

$$\Box a \land \Box b \dashv\vdash \Box(a \land b) \qquad\qquad \Diamond a \lor \Diamond b \dashv\vdash \Diamond(a \lor b)$$
$$\Box(a \lor b) \nvdash \Box a \lor \Box b \qquad\qquad \Box a \lor \Box b \vdash \Box(a \lor b)$$
$$\Diamond a \land \Diamond b \nvdash \Diamond(a \land b) \qquad\qquad \Diamond(a \land b) \vdash \Diamond a \land \Diamond b$$

In many-valued logic, one can define conjunction and disjunction respectively by *inf* and *sup*. This ensures in particular the classical behaviour of these operators. These notions refer to an order relation. We have two main possibilities for this order relation: considering a linear order and considering a partial order.

4.2. *Linear order and Łukasiewicz's nightmare*

Considering the following linear order between the four-values:
$$0^- \prec 0^+ \prec 1^- \prec 1^+$$
the table below describes the conjunction defined with *inf*.

$\land$	0^-	0^+	1^-	1^+
0^-	0^-	0^-	0^-	0^-
0^+	0^-	0^+	0^+	0^+
1^-	0^-	0^+	1^-	1^-
1^+	0^-	0^+	1^-	1^+

TABLE 7

We let the reader draw the table for disjunction defined with *sup*. We call LM4 a modal logic defined by such tables and TABLE 3.

THEOREM 4 *A LM4 obeys all axioms for a codi modal logic excepted the two negative ones.*

Proof. We prove that $\diamond a \wedge \diamond b \vdash \diamond(a \wedge b)$, and leave the other proofs to the reader. Suppose that $\diamond(a \wedge b)$ is 0, then $a \wedge b$ is 0^-, therefore a or b is 0^-. Suppose that a is 0^-, then $\diamond a$ is 0, therefore $\diamond a \wedge \diamond b$ is 0. Since we can do a similar reasoning when the value of b is 0^-, we can conclude.

That $\diamond a \wedge \diamond b \vdash \diamond(a \wedge b)$ holds is a serious drawback. This problem is in fact the nightmare Łukasiewicz had to face all his life. This is a central feature of his systems and he was not able to give a satisfactory explanation in order to justify it. The absurdity appears clearly through the following example:

If it is possible that it will rain tomorrow and it is possible that it will not rain tomorrow, then it is possible that it will rain and not rain tomorrow.

This paradox appears also when one considers a many-valued system where one value is interpreted as possible and a formula and its negation can have both this value, conjunction being defined by *inf*. This includes the cases where possible is a designated or non-designated values and the case where $\diamond a$ is not a connective in the object language.
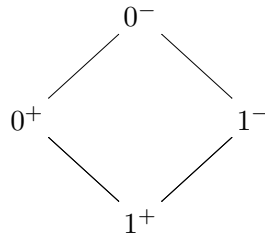
4.3. *The dream of partial order*

An option to avoid Łukasiewicz's nightmare is to use the following partial ordering:
$0^- \prec 0^+ \prec 1^+$
$0^- \prec 1^- \prec 1^+$
0^+ and 1^- are not comparable.
This can be represented by the following diagram:

The table corresponding to the definition of conjunction as *inf* in this context is the following:

$\wedge$	0^-	0^+	1^-	1^+
0^-	0^-	0^-	0^-	0^-
0^+	0^-	0^+	0^-	0^+
1^-	0^-	0^-	1^-	1^-
1^+	0^-	0^+	1^-	1^+

TABLE 8

We let the reader draw the table for disjunction defined with *sup* in this linear order. We call PM4 a modal logic defined by such tables and TABLE 3.

THEOREM 5 *A PM4 is a codi modal logic.*

Proof. We just prove that $\diamond a \wedge \diamond b \nvdash \diamond(a \wedge b)$, and leave the other proofs to the reader. Suppose that a is 0^+ and that b is 1^-, then $\diamond a$ is 1 and $\diamond b$ is 1, therefore $\diamond a \wedge \diamond b$ is 1. On the other hand $a \wedge b$ is 0^-, therefore $\diamond(a \wedge b)$ is 0.

5. *Negation*

We can define negation by the table below:

a	$\neg a$
0^-	1^+
0^+	1^-
1^-	0^+
1^+	0^-

TABLE 9

We call M4N a modal logic defined by TABLE 9 and TABLE 3.

THEOREM 6 *All De Morgan laws hold in M4N:*

Proof. We show that $\neg(a \wedge b) \vdash \neg a \vee \neg b$, leaving the other cases for the reader. Suppose that $\neg a \vee \neg b$ is 0, then $\neg a$ is 0 and $\neg b$ is 0, then a is 1 and b is 1, then $a \wedge b$ is 1, then $\neg(a \wedge b)$ is 0.

Examining the proof of this theorem, we see that the variation with + and - does not interfere, so that any definition of negation transforming designated values in non-designated values and vice-versa would lead to logics in which

all de Morgan laws hold, it is easy to check that such transformation also leads to the validity of the law of excluded middle $a \vee \neg a$ and the principle of non-contradiction $\neg(a \wedge \neg a)$ (whether we are using a partial or a total ordering).

Such option is different than the option of Dunn-Belnap four-valued semantics given by the table

a	$\neg a$
0^-	1^+
0^+	0^-
1^-	1^+
1^+	0^-

TABLE 10

leading to a logic in which all de Morgan laws hold but not the excluded middle, nor the principle of non-contradiction.

THEOREM 7 *The following laws hold in M4N:*

$\Box a \dashv\vdash \neg \Diamond \neg a$
$\Diamond a \dashv\vdash \neg \Box \neg a$

Proof. The table below shows that the first law is valid, we let the reader construct the table in order to show that the second law is valid.

$\Box a$	a	$\neg a$	$\Diamond \neg a$	$\neg \Diamond \neg a$
0^-	0^-	1^+	1^+	0^-
0^-	0^+	1^-	1^+	0^-
0^-	1^-	0^+	1^+	0^-
1^+	1^+	0^-	0^-	1^+

TABLE 11

It is easy to check that for the above theorem the simple inversion between designated and non-designated value is not enough.

6. *Implication and Kripke law*

Considering the standard definition of implication in terms of negation and disjunction, based on partial order, we have the following table

$\rightarrow$	0^-	0^+	1^-	1^+
0^-	1^+	1^+	1^+	1^+
0^+	1^-	1^+	1^-	1^+
1^-	0^+	0^+	1^+	1^+
1^+	0^-	0^+	1^-	1^+

TABLE 12

We see that, like in classical logic, $a \rightarrow b$ is 0 iff a is 1 and a is 0. This implication has in fact all the properties of classical implication, in particular we have: $\vdash a \rightarrow b$ iff $a \vdash b$.

We can formulate Kripke law as follows
$\Box(a \rightarrow b) \vdash \Box a \rightarrow \Box b$

We call PM4N a modal logic defined by TABLE 3, TABLE 9 and the tables for conjunction and disjunction based on partial order.

THEOREM 8 *Kripke law is valid in PM4N.*

Proof. Suppose $\Box a \rightarrow \Box b$ is 0, then $\Box a$ is 1 and $\Box b$ is 0. Since $\Box a$ is 1, a is 1^+. Since $\Box b$ is 0, b is either 0, in this case $a \rightarrow b$ is 0 and $\Box(a \rightarrow b)$ is 0, or b is 1^-, in this case $a \rightarrow b$ is 1^- as shown by the table above and so we have again $\Box(a \rightarrow b)$ is 0 .

Note that Kripke law is also valid in LM4N: the same reasoning can be performed using the table below where implication is defined using standard definition of implication in terms of negation and disjunction and considering disjunction defined with linear order.

$\rightarrow$	0^-	0^+	1^-	1^+
0^-	1^+	1^+	1^+	1^+
0^+	1^-	1^-	1^-	1^+
1^-	0^+	0^+	1^-	1^+
1^+	0^-	0^+	1^-	1^+

TABLE 13

7. *Necessitation and replacement*

We consider furthermore two famous laws for modalities:
Necessitation
   if $\vdash a$ then $\vdash \Box a$

Replacement

  if $a \dashv\vdash b$ then $\Box a \dashv\vdash \Box b$

  if $a \dashv\vdash b$ then $\Diamond a \dashv\vdash \Diamond b$

In LM4N, we have the following typical failure of the law of necessitation illustrated by the table below:

p	$\neg p$	$p \vee \neg p$	$\Box(p \vee \neg p)$
0^-	1^+	1^+	1^+
0^+	1^-	1^-	0^-
1^-	0^+	1^-	0^-
1^+	0^-	1^+	1^+

TABLE 14

From this example, we also see that replacement does not hold for $\Box$, since

  $p \vee \neg p \dashv\vdash q \vee \neg q$

but not

  $\Box(p \vee \neg p) \dashv\vdash \Box(q \vee \neg q)$

The counter-example given above for LM4N is not a counter-example for PM4N:

p	$\neg p$	$p \vee \neg p$	$\Box(p \vee \neg p)$
0^-	1^+	1^+	1^+
0^+	1^-	1^+	1^+
1^-	0^+	1^+	1^+
1^+	0^-	1^+	1^+

TABLE 15

We are therefore getting close to S5, especially if we consider a PM4N with reduction of repetition and composition. In a following paper we will study in details such logic showing that in fact it is a strict extension of S5.

8. *Further works and Acknowledgments*

We have seen in this paper how to construct many interesting modal logics using four-valued matrices, based on some intuitive ideas.

All the logics presented can also be axiomatized by sequent systems. This is straightforward applying many-places sequents (two places on the left for non-designated values, two places on the right for designated values), using a general method presented in [3]. It is also possible to provide bivalent

non-truth functional semantics for these logics. We will develop this topic in detail in another paper.

We will also show how we can apply many-valued tools to develop other modal logics: deontic, doxastic and epistemic logics, and also modal logics not based on classical logic such as paraconsistent modal logics.

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