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Opposition and Order

1. The hexagon of opposition and mathematical philosophy

The notion of order is one of the most fundamental concepts of mathematics. It is an intuitive notion that has been axiomatized in different ways. Our objective here is not to give a new axiomatization of it but to analyse this notion using the theory of logical opposition initiated by Aristotle, developed by Apuleius and Boethius into a square of opposition and more recently by Blanché into a hexagon of opposition (about history and recent works on the square of opposition, see Beziau and Payette, 2008, 2012, Beziau and Jacquette, 2012 Beziau and Read, 2014).

This kind of work can be seen as part of *mathematical philosophy* as described by Bertand Russell: “The other direction, which is less familiar, proceeds, by analyzing, to greater and greater abstractness and logical simplicity; instead of asking what can be defined and deduced from what is assumed to begin with, we ask instead what more general ideas and principles can be found, in terms of which what was our starting-point can be defined or deduced. It is the fact of pursuing this opposite direction that characterizes mathematical philosophy as opposed to ordinary mathematics.” (Russell 1919)

The notion of order is a very basic and primitive notion and it is no necessarily clear how we can go deeper without right away reaching a pure logical level, considering it as a binary relation from the point of view of first or second order logic and set theory. We will see here that theory of opposition is a useful tool for an intermediate understanding, not so abstract but yet logically articulated, that can be applied to order relation and also to binary relations.

Let us present the theory of opposition directly from Blanché’s hexagon which is an extension / improvement of the traditional square of opposition:

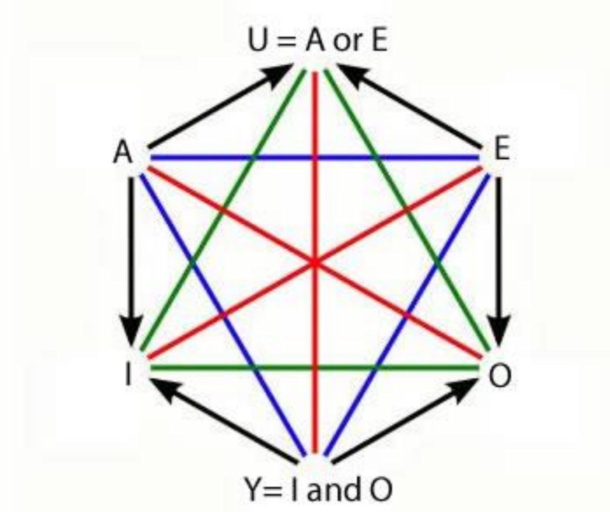

Figure 1: Abstract Hexagon

We have here the three notions of oppositions: contradictory (in red), contrary (in blue), subcontrariety (in green) and the notion of subalternation/implication (in black). Blanché's hexagon is the assemblage of a triangle of contrariety with a triangle of subcontrariety, tied together by contradiction, in which we find back the original square of opposition AEIO. We have used the traditional letter for the four corners of the square and Blanché's letters Y and U for the two additional vertices.

The vertices of the square and hexagon can be interpreted as propositions or concepts (embedded into propositions) and we remind here the definitions of the three notions of oppositions: two propositions are said to be *contradictory* iff they cannot be true and cannot be false together, *contrary* iff they can be false together but cannot be true together, *subcontrary* iff they can be true together but cannot be false together.

There are numerous applications/interpretations of the hexagon of opposition (see Beziau 2012a), in this paper our main interest is the application of this hexagon and related geometrical figures of opposition to order relations and connected relations.

2. Order hexagons

2.1. The total order hexagon of opposition

We start by presenting the following hexagon:

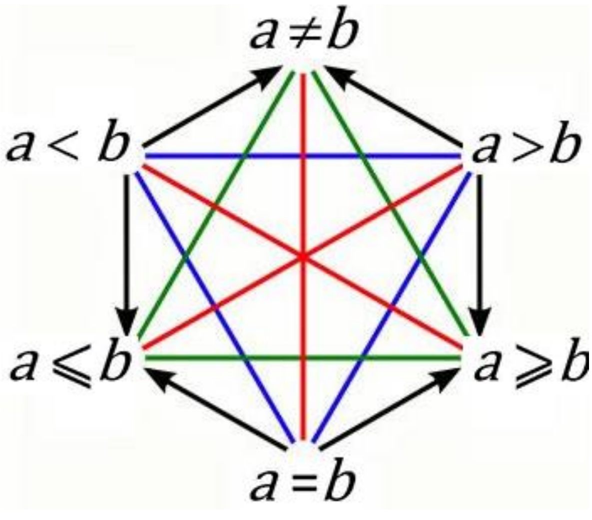

Figure 2: Total order hexagon

This hexagon is valid only in the case of a *total order* (also called *linear order*).[1] Let us remember the axioms:

[1] A quite similar hexagon was presented by Blanché (1966, p.64), but he didn't stress that it is limited to total orders, the same with Sesmat (1951, p, 412). For Blanché the fact that a (total) order relation can be described by a hexagon of opposition is the proof that this hexagon is not artificial since it reflects our most advanced conceptualization, i.e. mathematics. On the other hand we can argue that the hexagon is a deep structure that can give us a better understanding of mathematical notions,

(antireflexivity) $a \not< a$
(antisymmetry) if $a < b$ then $a \not< b$
(transitivity) if $a < b$ and $b < c$ then $a < c$
(totality) $a < b$ or $b < a$

The antisymmetry axiom can be deduced from antireflexivity and transitivity. The relation $\leq$ can be considered as an abbreviation, $a \leq b$ meaning $a < b$ or $a = b$. Taking this definition the axioms of a total order can be considered as follows:

(reflexivity) $a \leq a$
(antisymmetry) if $a \leq b$ and $b \leq a$ then $a = b$
(transitivity) if $a \leq b$ and $b \leq c$ then $a \leq c$
(totality) $a \leq b$ or $b \leq a$

2.2. The partial order hexagon of opposition

In the case of a partial order, an order defined by withdrawing the totality axiom, the contradictory of $a < b$ is not $b \leq a$ but is $b \leq a$ or $a \nparallel b$ which means $a \not< b$ and $b \not< a$ and $a \neq b$. In natural language, when $a \nparallel b$ we say: a and b are *incomparable*, and when $a \parallel b$ we say: a and b are *comparable*, which means $a < b$ or $b < a$ or $a = b$. The Y-vertex of the above order hexagon in case of a partial order is not $a = b$ but $a = b$ or $a \nparallel b$ and the U-vertex is not $a \neq b$ but $a \neq b$ and $a \parallel b$.

To avoid ambiguity between partial and total order it is common to use the following symbols for partial orders: $\prec$, $\preccurlyeq$. We can therefore draw the following hexagon for partial order:

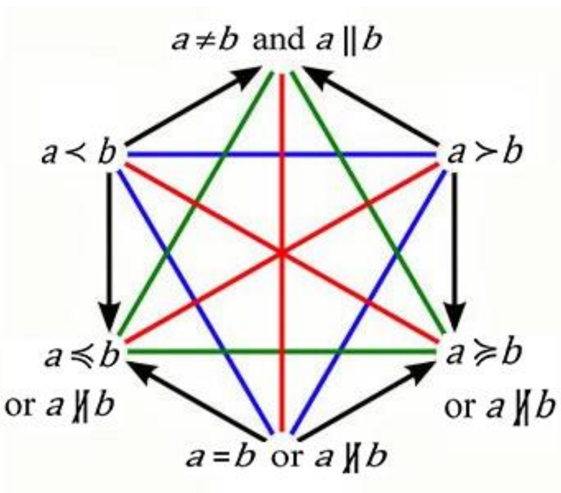

Figure 3: Partial order hexagon

3. Triangle, square, hexagon of sets

The most famous example of partial order is the relation of inclusion between sets. For sets we have the notations $\subset$, $\subseteq$ symbols analogical to those of order relations but there is no common terminology and notation to talk about the situation when given two sets none of them is included in the other. We can interpret the above axioms of partial order for sets, considering that $\prec$ is $\subset$, $\preccurlyeq$ is $\subseteq$ and $a \nparallel b$ means that neither the set a is included into the set b nor the converse. To transpose the terminology comparable/incomparable for sets would however be a bit strange, because for example two disjoints sets are incomparable but incomparable sets are not necessary separated in the sense they may have a common intersection.

3.1. Contrariety triangle of sets and theories

Anyway if we want to systematically classify the relation between two sets, the basic fundamental basic relations do not appear in this diagram. We can distinguish three cases: *inclusion*, *exclusion* and *intersection*.[2] By inclusion we mean here that a set is included into the other one or vice versa, by exclusion that the two sets are disjoint and by intersection that they have a non-void intersection without one of them being included into the other one. These three situations form a triangle of contrariety; they are exclusive and exhaustive: any pair of them are incompatible and there are no other possibilities.

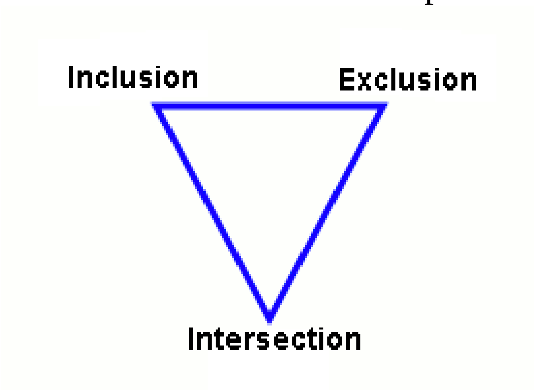

Figure 4: Contrariety triangle of sets

This classification is useful for conceptual analysis, i.e. if we consider that sets correspond to concepts. And on this basis it is possible to construct a deductive system similar to syllogistic (cf. Beziau 2014).

From this triangle, as in the case of any triangle of contrariety, it is possible to build a hexagon considering a dual triangle of subcontrariery whose three vertices are complementary pairs of each of the vertex of the triangle of contrariety:

[2] This trichotomy has been discussed by Blanché himself using this terminology (1966, p.67).

Contrariety Triangle	Subcontrariety Triangle
Inclusion	Anti-Inclusion = Exclusion or Intersection
Exclusion	Anti-Exclusion = Inclusion or intersection
Intersection	Anti-Intersection = Inclusion or exclusion

The problem of this subcontrariety triangle is that the three vertices do not really have a positive meaning – except the anti-exclusion vertex which corresponds to standard intersection (intersection that includes inclusion) – so this triangle is a bit artificial as well as the corresponding hexagon:

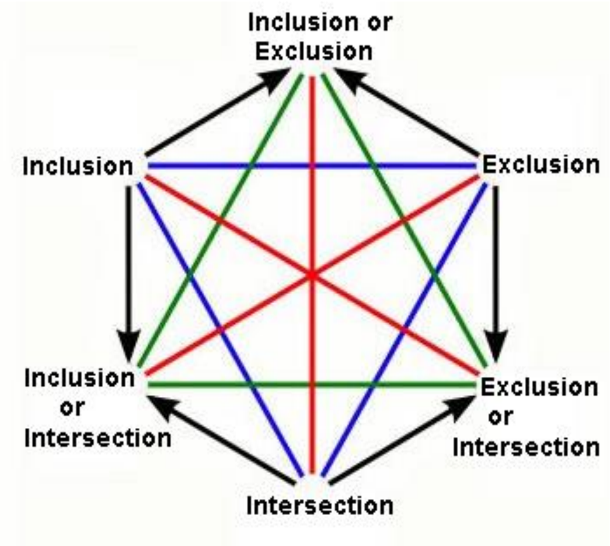

Figure 5: Hexagon of sets

A particular application of this geometrical articulation of sets is the consideration of sets of models of a formula or a theory. If all the models of a theory T1 are included in the models of a theory T2, we can say that T2 is reducible to T1, all what can be deduced from T2 can be deduce from T1. For example all the models of the theory of total order are models of the theory of partial order, all that can be deduced from the theory of partial order can be deduced from the theory of total order. If the intersection of the sets of models of two theories is empty we can say that the two theories are incompatible, in particular the conjunction of the axioms of these two theories is trivial, i.e. anything can be deduced from it. Finally if these two theories have a common model, we may say that there are compatible. In this framework compatible/incompatible is not a contradictory pair but a contrary pair which is completed with the notion of reducibility forming the following triangle:

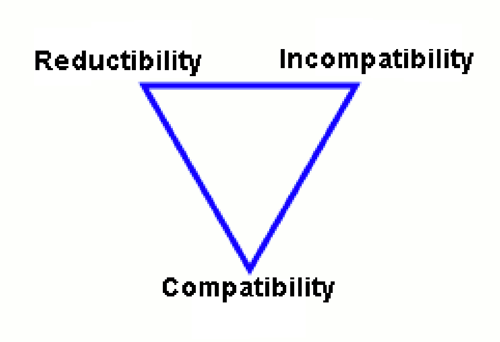

Figure 6: Contrariety triangle of theories

Here again the dual subcontrariety triangle and the hexagon that we can be built by assembling the two triangle do not make much sense.

3.2. Contrariety square of sets and truth-values

There is another kind of relation between sets that is interesting to consider which is connected with the notions of opposition appearing in the square and hexagon of opposition. Considering two sets within a given universe, we can consider the following situations: (1) *a* and *b* are disjoint and filling the whole universe, (2) *a* and *b* are disjoint but not filling the whole universe, (3) *a* and *b* are not disjoint but filling the whole universe, (4) *a* and *b* are not disjoint but not filling the whole universe. The first situation is dichotomy. The second situation is what corresponds to contrariety (this clearly appears if we consider that these sets are sets of models of propositions: there are models which are models of none of two propositions, this means that these two propositions can be both false). The third situation is subcontrariety (a similar remark as the previous one can be made). The fourth situation corresponds to none of the three oppositions, considering models of propositions, it means that the two propositions can both be true and can both be false.

This quadratic relation between two sets forms a square of contrariety that can be represented as follows (red means empty):

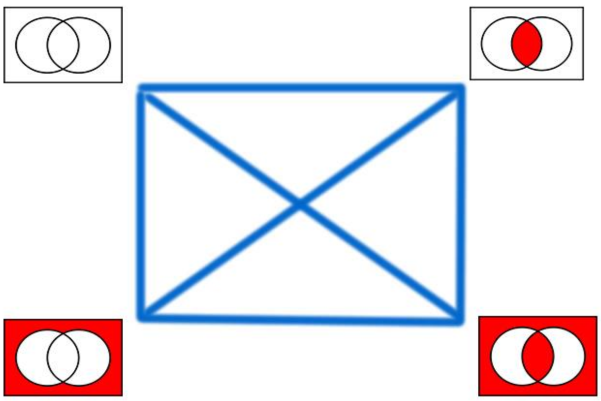

Figure 7: Contrariety square of sets

As in the case of a triangle of contrariety we can build a dual square of subcontariety and put together these two squares tied together by the relation of contradiction. From the geometrical point of view, following an idea of Moretti (2009) this construction can be improved by considering a simplex and a bi-simplex. But as in the previous case the dual figure and the resulting assemblage, being an octagon or a bi-simplex, are quite artificial.

Considering that two propositions can both be false and both be true is connected with the four-valued semantics of Dunn and Belnap. According to this semantics a proposition by itself can be can both true and false, or neither true or false. Defining negation in a certain way then a proposition and its negation can both be true and also both be false.[3] The truth-values form a square of contrariety:

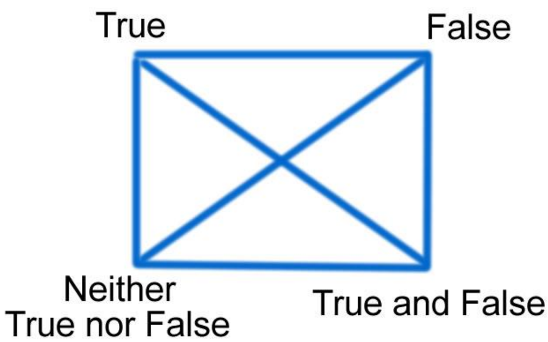

Figure 8: Contrariety square of truth-values

To find positive meanings of the vertices of the dual subcontrariety square and of those of the correlated assemblage is important to give more strength to this semantic framework.

4. Oppositional geometry from order to binary relations

4.1. Contrariety square of partial order

For relations of order we can also consider triangles and squares of contrariety. The hexagon of total order (Figure 2) is mixing identity with strict order and this makes sense to produce an elegant hexagon. In the case of the hexagon of partial order (Figure 3) which furthermore involves the notion of comparability this is not so nice.

In the case of a total order we have basically three possibilities considering the relation between two objects *a* and *b* which is depicted by the triangle of contrariety which is inside the hexagon. In the case of a partial order we have in fact four possibilities which are compressed into three with the Y-corner encompassing two of them (with an exclusive disjunction). Instead of this we can build the following square of contrariety:

[3] There are different ways to present such a semantics, for a discussion about this, see Beziau 2012b.

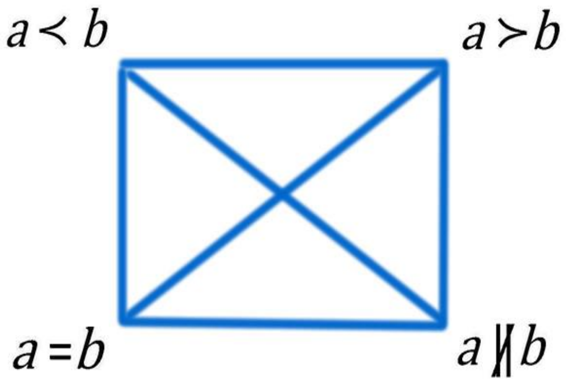

Figure 9: Contrariety square of partial order

From this we can build a dual subcontrariety square by taking the classical negation of each vertex, but again we don't have here clear positive meanings, although this construction specifies that for example the negation of equality, i.e difference, is the disjunction of strictly inferior, strictly superior and comparable. And in this case there is a name, the same for the pair incomparable/comparable. But if we consider the negation of strictly inferior, it is not clear how we shall name it, in a previous paper (Beziau 2012a) we have suggested "major" (and "minor" for the contradictory of strictly superior") presenting the following hexagon:

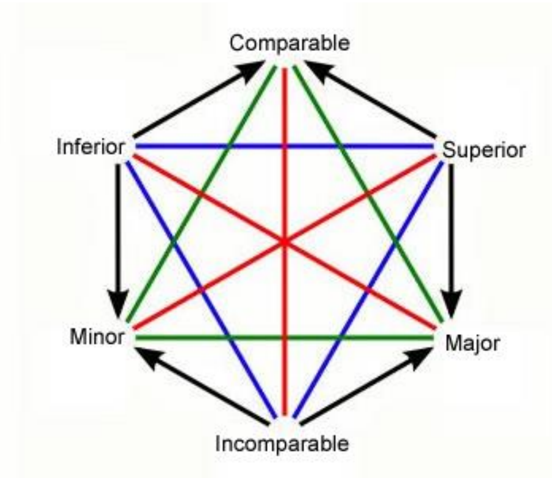

Figure 10: Simplified hexagon of partial order

This terminology is not completely satisfactory and this hexagon is a simplification in the sense that we have withdrawn the case of identity, this can be done if we consider that we are dealing by principle with objects which are different. But this simplification does not work in the case we are considering properties of things, like “the strength of”, because two objects may be different and having the same strength.

4.2. Contrariety square of antisymmetric antireflexive relations

The figure of oppositions we have presented in connection with order relation does not depend on transitivity, the important point is antisymmetry. Antisymmetry is in fact the starting point considering that $a < b$ and $b < a$ are contrary, i.e. they cannot be true together. If we consider any antisymmetric antireflexive relation $\prec$ we can draw the same diagram: we have the same four exclusive possibilities: relation in one direction, in the opposed direction, no relation at all, identity.

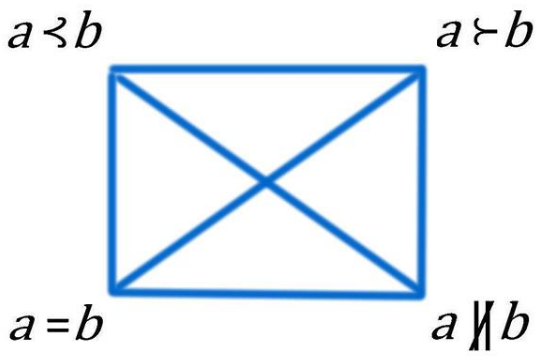

Figure 11: Contrariety square of antisymmetric antireflexive relation

4.3. Hexagon of binary relations

In the case of a relation which is only antireflexive, the pair $a < b$ and $b < a$. is not necessary a contrary pair. Besides $a = b$ and $a \not\parallel b$, we have $a < b$ or $b < a$, which means $a \neq b$ and $a \parallel b$. We have therefore the following hexagon of opposition.

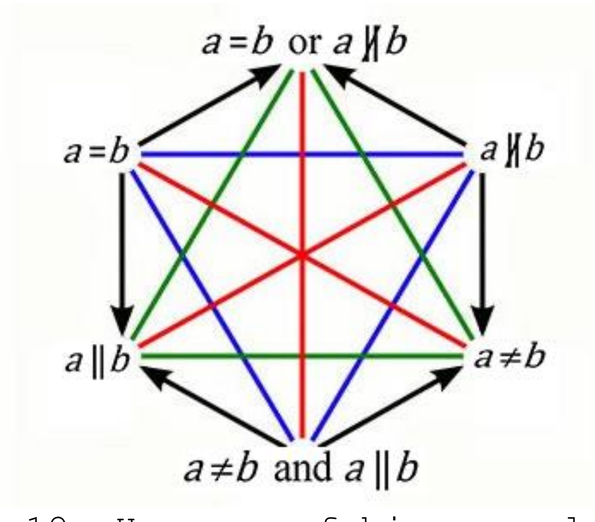

Figure 12: Hexagon of binary relations

We have called this diagram, the hexagon of binary relation, because the situation is exactly the same if the relation is not antireflexive, i.e. in case there is no specific axiom for the relation. Is this hexagon a perfect description of what is going on for a binary relation? Not quite.

In case of an antisymmetric relation R, $a\,R\,b$ and $b\,R\,a$ implies that $a = b$, but for a relation which is not antisymmetric, we can clearly distinguish these two situations. We can introduce the symbol ↓↑ to express the fact that $a\ R\ b$ and $b\,R\,a$ and $a \neq b$, and also we can distinguish this situation from the bottom vertex of the above hexagon introducing the symbol ‡↑ to express that fact that $a\,R\,b$ or $b\,R\,a$ but not a ↓↑b and also $a \neq b$, which simply means that the relation is holding in only one direction between two different objects. In this case it is better to use the symbol ‡⇞ for incomparibility rather than ∦. We have then the following square of contrariety:

Figure 13: Square of contrariety of binary relation

We can consider the dual subcontrariety square of which each vertex is a disjunction of three vertices of the above square, which don't really have positive intuitive meanings. We can reduce the above square to a triangle of contrariety by withdrawing the top right corner, considering that we are dealing with relations only between two distinct elements. Then we can draw a hexagon by assembling this triangle with the dual triangle of subcontrariety. In this hexagon only one vertex have a clear meaning, it is the negation of a ‡⇞ b which can be interpreted as "a and b are related".

As we can see from this study, the notion of contrariety is predominating, and we can say also that despite the fact that some squarse of contrariety appear, it seems that we can most of time think using a triangle of contrariety (which can be extended or not into a meaningful hexagon). We can use this analysis to support Blanché's idea that our thought is triangular (cf. Larvor 2009).

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