# Universal Logic

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"Un mathématicien, un mathématicien moderne en particulier, se trouve, dirait-on, à un degré supérieur de l'activité consciente: il ne s'intéresse pas seulement à la question de *quoi*, mais aussi à celle du *comment*. Il ne se borne presque jamais à une solution—tout court—d'un problème, il veut avoir toujours les solutions les plus...les plus quoi?—les plus faciles, les plus courtes, les plus générales, etc."

A.Lindenbaum

**Abstract**

Universal Logic is a general study of logic in the same way as Universal Algebra is a general study of algebra. It is based on the fact that there is no One Logic or Absolute Laws of Logic, but rather a type of logical structures who are fundamental mother structures in the sense of Bourbaki. Logic is then an autonomous field of mathematics, with its own intuitions and concepts and which can survive and be developed without importing specific notions from other fields of mathematics.

After criticizing the Polish approach, we give a definition of logical structures and we show what kind of work can be done within this framework studying the abstract form of the completeness theorem. A central notion is emerging, the concept of excessive theory, which is the start of a new "boumologie".

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**Contents**



## 1. What is Universal Logic?

### 1.1. The religious stage

During this century we have been witnessing the birth of a non denumerable infinity of new systems of logic. Some people think that most of these systems are meaningless formal mind games, which have importance only for those who are toying with them. Some people say that each system has its own value and importance, that "classical logic is wrong", that if we want to buy cigarettes, or to play with very small objects, e.g. *des petits grains de poussières*, we cannot use it.

Are there many logics or only one? and what is Logic?

Traditionaly, logic was conceived as the science of the *laws* of reasoning: in the same way as the bird in the sky is flying according to certain rules, our reason is moving according to a determined pattern (see e.g. the *Logik* of E. Kant, Introduction, §1 *The concept of logic*).

Following this scheme we can look for the absolute laws of thought in the same way as we are looking for the general equation of the universe which will explain everything, give the key to the mystery of the world.

This attitude seems rather naïve and primitive, and according to the terminology of Auguste Comte belongs to the "religious phase" of mankind, that is to say, the first and less developed stage of human being.



We can jump from absoluteness to relativity, saying that they are no absolute laws but relative laws and various logics, but if we stay at this level we are still at the "religious stage": we have just swapped monotheism for polytheism (and this is not necessarily progress).

### 1.2. The example of Universal Algebra

We can give a more precise account to the rejection of the law-approach, based on the development of Universal Algebra.

In Algebra we can distinguish three levels relatively to the degree of abstraction (see Figure 1). In the more concrete case an algebraic structure is a structure with specific elements, according to the nature of these objects, certain laws are valid. A first step in abstraction makes us forget the *nature* of the objects and we keep only the laws they obey, then we are into *Abstract Algebra*.

The second step in abstraction, leading to *General Abstract Algebra* or *Universal Algebra*, was taken by G. Birkhoff, when he decided to consider as an algebra any set with operations; as he explained (see e.g. his paper of 1946), this was the only way of reaching unity among all algebras. This means that, this time, abstraction of the laws has been realized and we are left with any kind of functions. It may seem much too general. But Birkhoff succeeded in proving a lot of interesting results (in his famous papers of 1933 and 1935); that was not the case of A.N. Whitehead who wrote the first monograph on Universal Algebra forty years earlier (1898) but who, according Grätzer (see his monograph, p.*vi*), "had no results", though he "recognized the need for universal algebra".

In fact what Whitehead called Universal Algebra will better be called today only Abstract Algebra. He was still at the second level considering as absolutely valid an algebraic law such as the law of associativity. But his book is very interesting from the point of view of the philosophy of mathematics because he anticipated the idea of universal algebra by his search for generality, and therefore has strongly contributed to its emergence.

**The Way to Universal Algebra**

*The three levels*

| | Abstraction |
|---|---|
| <u>General abstract algebra</u><br>A Groupoid<br>$\mathcal{A} = \langle \mathbf{A};\gamma \rangle$<br>$\gamma$ is a binary function on **A** | Second level of abstraction: abstraction of the laws |
| <u>Abstract algebra</u><br>A Group<br>$\mathcal{G}=\langle \mathbb{G};* \rangle$<br>* is an associative operation with neutral object and symmetric | |
| <u>Specific algebra</u><br>The Group of Integers<br>$\mathcal{Z}=\langle \mathbb{Z};+ \rangle$ | First level of abstraction: abstraction of the elements |

Figure 1

Remark: if we go further and drop functionality we are at the level of the general theory of structures.

It appears that thc lesson of Universal Algebra is that:

- there is no One Algebra who will be the Queen of Algebra,
- there is no Absolute Laws of Algebra which will rule the whole Kingdom of Algebra for eternity, and even for the present time.

Now let us point out the main features of the Universal Approach which appear in Universal Algebra and which will be our guides for Universal Logic:

- unity
- generality
- abstraction
- undetermination

We say that Universal Logic is a general study of logical structures in the same way as Univeral Algebra is a general study of algebraic structures. In particular there will be no Paramount Logic, such as Classical Logic, Intuitionistic Logic, or whatever Polar Logic,[1] and no Absolute Law of Logic, such as the law of contradiction, excluded middle, identity, or whatever categorical law from the sky.



To explain exactly what Universal Logic is we must therefore explain what is a logical structure, this will be the task of the next section.

## 2. The Architecture of Mathematics

### 2.1. Logic and Mathematics

The relations between logic and mathematics are multiple and ambiguous. For example, is *mathematical logic* the logic of mathematics or the mathematic(s) of logics?

This dual aspect is well represented in the history of modern logic, with on one hand the Fregean approach and on the other hand the Boolean approach.

The Fregean project was not only to study the logic of mathematics but to reduce mathematics to logic, and for him logic was not exactly mathematics but rather a symbolic representation, a language more precise and more perfect than the usual one.

At the opposite Boole tried to give a mathematical account to the laws of thought in general (not only of the mathematical thinking), expressing them in a way similar to the laws of algebra.

One can think that the Fregean approach is not sufficiently mathematical because his "language" is not sufficiently mathematical, and one can think that the Boolean approach is not sufficiently mathematical because it cannot be used for the description or the foundations of mathematics, as Louis Couturat wrote at the end of *L'Algèbre de la Logique* (1905): "One can say that the Algebra of Logic is a *mathematical* Logic, by its form and method; but one must not consider it as the Logic of *Mathematics*."

If one is interested in the logic of mathematics rather than in the mathematic of logics, the Fregean approach sounds better, that's why Frege rather than Boole is considered as the Father of modern logic, because he is taken as the founder of first-order logic,[2] and, on the contrary, Boolean logic is assimilated to propositional calculus (J.Van Heijenoort, especially, has contributed to the deification of Frege and to the rejection of Boole, see e.g. his posthumous paper of 1992).

But in fact it is misleading to reduce Boole to propositional calculus. It appears that the Boolean project was very important and not only for logic, but also for mathematics in general. Boole has strongly contributed to the development of Universal Algebra and to the modern conception of mathematics. As Whitehead says in his book on Universal Algebra, Boole was the first to consider that the laws of algebra are not only concerned with numbers or quantities, and this led to the notion of abstract algebra and furthermore to the notion of abstract structure where the nature of mathematical objects are undetermined. Thus Boole was one of the precursor of the notion of mathematical structure, concepts without which first-order logic is merely a game for blind people.

And if one is interested in the mathematic(s) of logics the Boolean project seems much more appealing that the Fregean project, even if one is interested only in the logic of mathematics, because the logic of mathematics is worth studying mathematically, that is to say in the spirit of mathematics which is probably not only a language, but rather the study of abstract objects given and shaped by structures.

This mathematical approach to logic has been developed especially in Poland. Lindenbaum considered that the Fregean-style language for the propositional calculus was in fact a mathematical structure, an absolute free algebra, and then Tarski reduced the Fregean-style propositional calculus to a Boolean algebra (a construction known as a Lindenbaum algebra or Tarski-Lindenbaum algebra).[3] From this point of view it seems that what was a drawback was not the Boolean approach (as J. van Heijenoort claimed in the paper already referred to), but the Fregean one.

A lot of things have then been worked out which have been collected in the book of H. Rasiowa and R. Sikorski with a suggestive name: *The mathematics of metamathematics* (1963). In fact this mathematical approach to classical logic was naturally extended to other logics, mainly because the mathematical concepts used for the study of classic logic were easily adaptable to study other logics, if not a suggestive everlasting source of birth for new logical systems. And following the book of 1963 there was *An algebraic approach to non-classical logics* by H. Rasiowa in 1974.

Some people may criticize this orientation of mathematical logic saying that it is rather algebra than logic, and they are not necessary wrong. The problem is to know whether or not logic can stand by itself as an independent part of mathematics.

We will argue that logic must be an independent mathematical field, that it needs to have its own mathematical concepts which are not necessarily the same as those of other mathematical branches. For example there is no reason that the notion of ultrafilter should be a key concept of logic. In §4 we will show that in fact it *cannot be* a key concept of logic and that a different key concept is emerging in logic: the concept of *excessive theory*.

In fact the defect of an algebraisation of logic already appeared in the work of Boole who proceeded to identify "or" with the addition, for this he was criticized by Jevons (Jevons was right and today the addition in a Boolean ring is interpreted as the symmetrical difference and the disjunction is only an indirect operation defined from the other operations). This illustrates an important point: if we are not keeping contact with the basic intuitions of logic and we are applying any mathematical tools already well-working in other fields of mathematics we will get something which is not what we sought (this recalls the famous story of the man looking for his lost key not where he lost it but where there is light). In a preceding paper (1994b) we have pointed to one example of this abuse: it seems nice, from an algebraic point of view, that the replacement theorem holds in a logic, thus many people are studying intensional logics in which the replacement theorem holds, without paying attention that this mathematical property is antinomic with the concept of intensionality.



Of course it is always illuminating to do some importing-exporting between different mathematical fields, but if this is to be valuable, the different fields must be really different and must have their own concepts and intuitions.

Universal Logic as the mathematic of logics may appear disgusting for those preoccupied with the problem of the foundation of mathematics, which is not playing a "fundamental" role in Universal Logic, if it is playing a role at all. For example, J. Porte in his book *Recherches sur la théorie générale des systèmes formels* (1965) which is written in the spirit of Universal Logic, writes (p.2): "A lot of people consider mathematical logic as the study of the philosophical problem of `the foundation of mathematics'. This problem will not be studied, I will even not discuss the question of whether it has a scientific meaning or not."

However we must emphasize that Universal Logic has an interest for the philosophy of mathematics and the foundation of mathematics but in a sense which has nothing to do with the logicist, formalist or intuitionistic approaches, and the logically oriented philosophy of mathematics.

Foundation of mathematics does not necessary means consistency and the reduction to a minimal formal linguistic system. It can also be taken as the understanding of what is the nature of mathematics. Universal Logic will show us, from within (like Universal Algebra also does), as a part of mathematics, how such kinds of process so "fundamental" in mathematics, as those of abstraction and generalization, work.

### 2.2. Structures: Species and types

Now let us see how Logic can take its place within mathematics.

According to Bourbaki, Mathematic is the study of mathematical structures, and there is no "s" at the end of the word because he thinks that Mathematic is not a random collection of various different things but that it has an "Architecture" (see his famous paper of 1948).

There are three basic mother-structures: algebraic structures, topological structures, structures of order. All the other mathematical structures can be constructed as "cross-structures" from the fundamental structures.

The basic structures are in fact very general, they just represent a "way of thinking", there is an algebraic way of thinking, a topological way of thinking, an order-like way of thinking. A way of thinking is connected to various intuitions and representations which are shaped into concepts. At this level those are rather undetermined concepts. Then, when we go down for complexity, we reach specifie well-determined structures.

A specific class of structures, like for example Boolean structures, can be reached in various ways. Each way is a different way of looking at the same class of structures. The class itself, independently of the way of looking at it, will be called a species of structures. By opposition the notion of type is connected with the way of looking.

Let's take another example, the *Axiom of Choice* can be expressed in many ways, each formulation has a certain type, and the thing they all refer to is the species. The Axiom of Choice in its Kuratowski-Zorn's formulation has an order-like type, in its Zermelo's formulation it has a function-like type. Each formulation is based on different intuitions and concepts. And to prove the equivalence of two different formulations is not necessarily trivial.

To consider species independently of types is "typical" of the extensionalist approach. But at the "fundamental" level, the extensionalist approach is meaningless, because we are left with types. The different ways of looking do not reveal the same things, because there is no categoricity or completeness. Thus they turn to be more important than the things they are revealing. They are like telescopes or microscopes which give partial information and that we can use to reach determinated partial informations about all kind of things. This is like seeing everything in black or white or in red and blue.

Therefore we can see that they are really two different approaches in the "foundations of mathematics" which have totally different meanings: the extensionalist one, set theory, and the intensionalist one, category theory (which is not at all eoncemed with "catagoricity"). Of course it is possible from category theory to go down to determination, but this not very interesting. On the other hand it seems that from set theory we cannot go up to undetermination, although the notion of set we are using in everyday mathematics is rather undetermined, in the sense that most mathematicians don't need to know a lot of set theory because they are using it rather as a general conceptual framework



than as a fundamental axiomatic basis from which everything on earth can be derived.

**Architecture of Mathematics**

General Theory of Structures

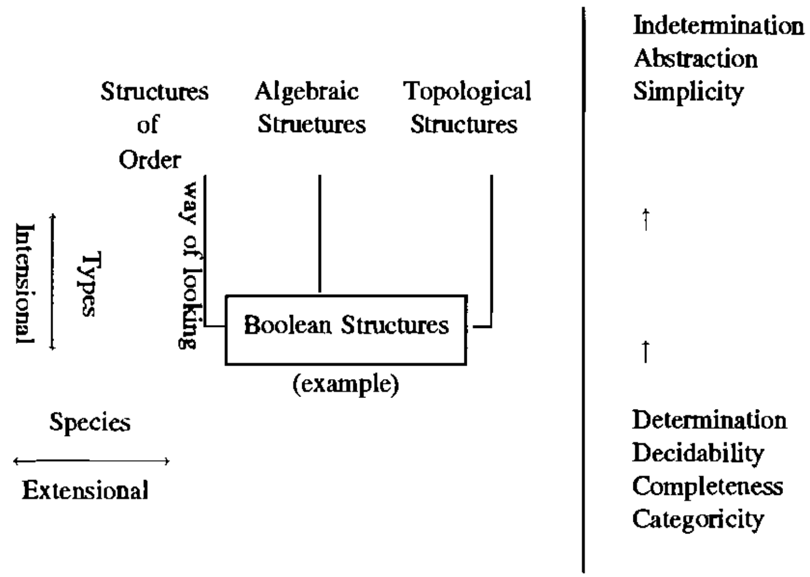


Figure 2

Within this description of the Architecture of Mathematics we will consider Logic as a class of fundamental structures.

This means that Logic corresponds to a way of thinking which is different from the other three fundamental ones, that there are typical mathematical concepts for logic which are related to typical ideas and intuitions.

### 2.3. The Ontology and Philosophy of Modern Mathematics

According to the ontology of modern mathematics, mathematical objects are objects of structures. What they are depends on the structure of the structure, that is to say on the relations of the structure they are merging in. To know what a given object is, is to know how it is related to the other objects of the structure.

This approach is very illuminating, especially for those who are engulfed in existential torments searching for answers to questions such as: "what is a number?", "what is as set?".

Now this approach will also be radical as regards those never-ending scholastical discussions about *propositions, sentences, formulas, statement*, etc. To answer the question "What are logical objects?" is to provide a logical structure.

All this means also that the philosophy of logic and the philosophy of mathematics are not necessary connected with the so-called philosophy of language or the so-called analytic philosophy, but seem rather closer to the most abstract metaphysics of Plato or Descartes as Lautman as strongly emphasized (see his doctoral dissertation of 1938).

## 3. Logical structures

### 3.1. The Polish approach,

The first tentative of development of a theory of logical structures was initiated by Tarski and was pursued mainly in Poland; it is known as the theory of the consequence operator.

A *structure of consequence* is a structure $\mathscr{C}=\langle\mathscr{A};\mathrm{Cn}\rangle$ where:

- $\mathscr{A}$ is an absolute free algebra $\langle \mathbf{A}; f_{i\,(i\in I)}\rangle$
- Cn is a function from $\mathscr{P}(\mathbf{A})$ to $\mathscr{P}(\mathbf{A})$ obeying the following four axioms:

(1) $A \subseteq \mathrm{Cn}\, A$,

(2) $A \subseteq B \Rightarrow \mathrm{Cn}\, A \subseteq \mathrm{Cn}\, B$,

(3) $\mathrm{Cn}\,\mathrm{Cn}\, A \subseteq \mathrm{Cn}\, A$,

(4) $\sigma\, \mathrm{Cn}\, A \subseteq \mathrm{Cn}\, \sigma\, A$, for every endomorphism $\sigma$ of $\mathscr{A}$.

**Critics**

1) We can criticize the *type* of the structure, which looks like a topological type. This can be an advantage if we want to apply topology to logic but it is a defect if we want to develop logic by its own.

2) We can criticize the *laws* for the consequence operator. Tarski chose the laws (1), (2), (3) because he had in mind the deductive relation given by what we called today Hilbertian systems of deduction. It is well known that a relation of deductibility induced by such systems obeys the laws (1), (2), (3). But if we take a more general notion of systems of deduction, like Gentzen's systems, they don't necessary hold in general.



On the other hand it is easy to find intuitive reasons to reject each of these laws, based on concrete examples. And even without any good counter-examples we can reject them in view of generality: *logic has no better reason to be monotonic than algebra has to be commutative.*

This does not mean that we must rejeet them absolutely, but that basic concepts must not depend on them. And furthermore we must not take them as a whole, but we must analyse them separately.

3) Finally we can criticize the underlying structure $\mathscr{A}$ (and the axiom (4) which is connected with it). In fact this additional aspect was not here at the beginning of the creation of Tarski and was put in only after the war by Łoś and Suszko (but this addition was very natural and quite inevitable given that the concept of matrix was already vivid during the 30s in Poland). This aspect is what is called "structurality". Given the underlying structure $A1, it seems very natural to put the axiom (4). In fact structurality is a perfeet expression of the idea that logical truth doesn't depend on the content but is only a question of form. Łoś and Suszko (see their famous paper of 1958) realized that substitutions were exactly endomorphisms, and give then a elegant and powerful treatment of what is generally treated confusedly as a "language", with such notion as concatenation. They have cleaned logic of linguistic accretions and that was badly needed. For example, in this light, it appears that what is called "Polish writing" is not a "writing" but the simplest (mathematical) representation of an absolute free algebra.

How ever nice the concept of structurality is, it seems better to reject it at the most general level, because there are some logics which are not structural (cf the quasi-formal logics of J.-L. Destouches and P. Février) and because this concept can be attacked for philosophical reasons by someone who is not happy with this song-and-dance form/content approach. But the very reason is that the difference of generality between the study of logics with specified underlying structures and without is enormous.

We call *Abstract Logic* the approach of logic which doesn't take in account the underlying structure (thus we don't use this word in the same sense as Suszko). Some concept are abstract, some are not. The idea is to work at the abstract level, then we will have some results which will be valid for all "languages", infinitary or not, structural or not, propositional or not.

We think that we can deal only *with structures, without structurality.*

In fact the structural approach is closely connected with matrix theory and thus with algebra. From this point of view it is also a reductionist approach to logic. Then it is better to first develop independent and autonomous concepts at the abstraet level and to see, *after*, how we ean put an underlying algebraic-like underlying structure. Then it will be interesting to see how our typical logic concepts can absorb a part of another field of mathematics. But they will not have necessarily to do so, because there are some logics to which it is meaningless to apply matrix theory. Then the algebraic treatment is just concerned with structurality. This is the case of various systems of N.C.A. da Costa. What is being used in this case is the *Theory of Valuation* of da Costa which is a more general approach, directly connected to abstract concepts (see da Costa/Béziau, 1993, 1994).

### 3.2. A definition of logical structures

Now we give the following definition of a *Logic* (or *Logical Structure*, or *Abstract Logic*),

A *Logic* $\mathscr{L}$ is pair $\langle \mathbb{L};\vdash\rangle$ where:

- $\mathbb{L}$ is a set,
- $\vdash$ is a relation over $\mathscr{P}(\mathbb{L})\times\mathbb{L}$ (ie, the cartesian product of the power set of $\mathbb{L}$ and $\mathbb{L}$).

The *quality* and *quantity* of the elements of $\mathbb{L}$ are left unspecified.

What about given them a name?

Well, that's not easy. Of course we will not call them "formulas", because this is a very insidious name. It is connected with the distinction of form/content and thus, with structurality. It has also linguistic connotations, a formula refers to a string of signs, generally unintelligible.

In fact it is not necessary to give them a name at the abstract level, the element of a topological space are called "points", but there is no name for elements of an abstract algebra, because what is primordial is not the nature of the elements but the structure there are merged in, thus in the case of algebra, what is important is the name "function", or "operation".

In logic what will be fundamental is the relation $\vdash$, the intuitive idea we get of it, and how we will call it. We will not call it a "consequence relation" because the word "consequence" is already used in the Polish approach. Some use such word as "entailment", or "derivability", "inference" which are rather cumbersome and ugly. We will use the name "deducibility", and the expression "relation of deducibility", because it is not erroneous to consider logic as the "theory of deduction".

There are, at least, two criticisms of this name. First some people will say that logic is not only deduction but also induction. Second some people will say that deduction is discrete and finite, identifying it with the notion of proof.



We can say that $T\vdash a$ means that there is a way leading from $T$ to $a$. In general, at the abstract level, we don't know which way it is and we don't need to know. But we have no reason to suppose that this way can be traced by a computer, or by any kind of mechanical process.

Finally it will be useful to give a specific name to subsets of $\mathbb{L}$, so we will call them *theories*.

### Criticisms of the proposal

Numerous criticisms can be formulated, we shall just select a few and give them an answer.

1) This definition is too much general. This is the most general criticism that could be made, and we could simply answer that it bears the defect it is pointing out. But let us be fair. Of course our definition includes a lot of parasites which are not and will never be logics in any reasonable sense of the word. But we can go on, without being preoccupied with that. The example of Category Theory just shows that it is not important. How many categories are not categories of structures? We can say that from the instant that the basic idea has been caught, Undetermination is better than Overdetermination.

2) This definition is not enough general. For example $\{a,a\}\vdash b$ identified with $\{a\}\vdash b$. Thus non-contraction logics are not included.

There are various ways to rectify this defect. One is to develop our theory of abstract logic within the framework of non-extensional set theory (in the sense that the axiom of extensionality is not valid).

Another way will be to consider sequences of elements of $\mathbb{L}$ rather than subsets of $\mathbb{L}$, or other structured subsets of $\mathbb{L}$. This solution, which seems the simplest, is however not more general, and we can consider this kind of structures as a particular case of abstract logics.

3) This notion is equivalent to the notion of structure of consequence.

Yes and No. Yes, because for every structure of consequence there is one and only one logic and vice-versa. No, because they are not of the same *type*. This equivalence will turn to be an advantage because then the notion of structure of consequence will be a bridge between logic and topology.

4) Why not taking another structure? A neighbourhood structure, like a multi-conclusion relation, i.e. a relation on $\mathscr{P}(\mathbb{L})\times\mathscr{P}(\mathbb{L})$?

This question can be answered on philosophical basis, saying that what it is important is to know which proposition can be deduced from which set of hypotheses. We don't need to know *all* of what can be deduced. The generalities we need are caught by properties of theories: completeness, consistency, closureness, etc. That why the multi-conclusion approach seems superfluous.[4]

J. Porte in his book of 1965 has studied various kind of structures, and his work is very instructive. Of course, at the beginning, there are necessary some *tâtonnements*, the experience and the time will show what is the best solution.

Thus the best way to stop critics is to show how our definition works and that it works well, this will be our preoccupation in the next section.

## 4. Univeral Logic, an example: Abstract Completeness

Abstraction and generalization can be considered by the mathematician as the supreme virtue or as the worse defect. Abstraction can lead to enlightment and best understanding it can also lead to General Abstract Nonsense, triviality and madness.

Our task in this section will be to show that our abstractions are good abstractions.

But first we must say that all this stuff of abstract logic is not a made-in-νεφελοχοχχυγια product, it has emerged progressively for a wide range of logical systems which have been studied and also from various attempt to systematize them.

In this section we will show how Universal Logic can be used with regard to completeness theorem, which is without doubt a central theorem for all logics. As this paper is rather expository we shall not enter into very technical details, which anyway have been worked out in some other papers refered.

### 4.4.1. Valuation as homomorphims/Valuations as maximal theories

There are two ways to look at a bivaluation of the classical propositional logic. First, a *bivaluation is an homomorphism* from the set of propositions, taken as an absolute free algebra $\mathscr{P} = \langle \mathbb{P}; \vee, \wedge, \rightarrow, \neg \rangle$, to an algebra $\mathscr{B} = \langle \{0,0\}, f\vee, f\wedge, f\rightarrow, f\neg \rangle$, which is equivalent to the Boolean algebra on $\{0,1\}$. The set of bivaluations is the set of homomorphisms from $\mathscr{P}$ to $\mathscr{B}$. It is possible to consider equivalently the set of functions from the set of generators *ATOM* of $\mathscr{P}$ to $\{0,1\}$ because it is a property of an aboslute free algebra that each function from the set of its generators to the domain of an algebra of the same type as a unique extension which is an homomorphism.



Second, *a bivaluation is the characteristic function of a maximal theory*, and the set of bivaluations is the set of characteristic functions of maximal theories.

This second way of looking at a bivaluation is more general and more abstract. *More general* because not every logic has a "homomorphic" semantic, *more abstract*, because the notion of maximality does not depend on the subjacent structure, the so-called "language".

What is the definition of a "maximal" theory? There is a definition which is not an abstract one, depending on the notion of negation, but this definition can easily be put in an abstract shape. Let us say that a *theory is limited* if and only if it is not possible to deduce everything from it; we call **LIM** the class of limited theories of a given logic, then a maximal theory is a maximal object with regard to the structure of partial order $\langle \mathbf{LIM}; \subseteq \rangle$.

Imagine that we can prove that for every logic, the set of (characteristic functions) of maximal theories is a sound and complete semantic, then we would be able to apply this general theorem in each particular case: checking if a set of bivaluations is the set of (charateristic functions) of maximal theories.

However as we will see it turns to be that the concept of "maximalility" is not the good one, and that we need another concept which is purely logical, by opposition to the concept of amaximal set which is quite the same as the concept of ultrafilter.

### 4.2. The most abstract form of the completeness theorem

A theory $T$ is said *a-excessive* if and only if it is $a$-limited (i.e. $T \nvdash a$) and for every $b$ not in $T$, $T, \cup \{b\} \vdash a$.

Given a deducibility relation $\vdash$ with the two following properties:

[Monotonicity] If $T \vdash a$ and $T \subseteq T$ then $T \vdash a$.

[Compactness] If $T \vdash a$ there exists a finite subtheory $T_o$ of $T$ such that $T_o \vdash a$

we have the following theorem:

LINDENBAUM-ASSER THEOREM *Every a-limited theory can be extended in an a-excessive theory.*[5]

From this theorem, we have a completeness theorem for the set of bivaluations of characteristic deductive functions (the *characteristic deductive function* of $T$, $\delta_T$ is defined as follows: $\delta_T(a)=1$ iff $T \vdash a$) of excessive theories (a theory is *excessive* if it is $a$-excessive for an object $a$).

COROLLARY: COMPLETENESS OF EXCESSIVE SEMANTICS

Proof. If $T \nvdash a$ then there exists an a-excessive extension $E$ of $T$. Let $\delta_T$ be the characteristic deductive function of $E$. $\delta_T(T)=1$ because $E$ is an extension of $T$, and $\delta_T(a)=0$ because $E \nvdash a$.

Remark. Some people will wonder what kind of completeness we are talking about, especially because we are not presenting here any "system of deduction" (such kind of thing with rules and proofs). But we must recall that systems of deduction are special cases of abstract logics.

### 4.3. The emergence of the concept of "excessivity"

Now we will explain why the concept of maximality is not the right concept and why the concept of excessivity is the good one.

We have proved (see our paper 'Excessive Theories') that the semantic of excessive theories is a minimal complete semantic, that is to say that a smaller semantic (a class of theories strictly included in the class of excessive theories) cannot be complete. As it is easy to see that maximal theories are excessive theories, this means that in the case where they are excessive theories which are not maximal then the semantic of maximal theories is not complete.

Of course it is possible to think that in all "good" logics all excessive theories must be maximal theories, and that was what most of the people were thinking. That's why this notion of excessivity, introduced by G. Asser a long time ago, has never been taken seriously, until some people in Brazil were trying to use the theory of valuation to give a semantic to logics in which there are excessive non-maximal theories.[6] In fact, it is the case of intuitionistic logic. We even have proved (in 'Excessive Theories') that if the law of Curry ($T, \neg a \vdash a \Longrightarrow T \vdash a$) was not valid in a monotonic compact logie with a minimal negation, then there were some excessive non-maximal theories.[7]

Suszko speaking about sets of bivaluations for a given inference relation is writing: "The adequate sets $V$ form an interval ($V_1 \subseteq V \subset V_2$) between the smallest adequate set $V_1$ and the largest one $V_2$. Some are better, some are worse." (in his paper of 1977, p.378).

Our "minimal result" shows that:

1) the adequate sets are generally not a linearly ordered,
2) there is no smallest adequate set in general,
3) on certain general conditions, there is a minimal adequate set, the set of excessive theories, which is, thus, not the worse.

It also shows that the version of Lindenbaum's theorem saying *every* a-*limited* *theory can be extended in a maximal theory from which* a *is not deducible*,[8] works only when all excessive theories are maximal, and then it is identical to the Lindenbaum-Asser theorem.



The concept of excessivity has plenty of good qualities. A powerful application is its application to sequent calculus.

We have shown (in 'Excessive Theories') that excessive theories preserve the rules of structurally standard systems of sequents.

From this and the theorem of Lindenbaum-Asser it results that we can prove immediately the completeness of a wide range of logics not necessary truth-functional and not necessary Fregean (i.e. where the theorem of replacement does not necessary holds).

For example, let's take a logic with a connective *, called supernatural implication, defined by the following rules of sequent calculus (we use ➡ in the same way as Gentzen):

$$\frac{\Gamma \Rightarrow a, \Delta \qquad \Gamma', b \Rightarrow \Delta}{\Gamma, \Gamma', (a*b) \Rightarrow \Delta, \Delta'}\ *l \qquad\qquad \frac{\Gamma \Rightarrow a, \Delta \qquad \Gamma' \Rightarrow b, \Delta}{\Gamma, \Gamma' \Rightarrow (a*b)\ \Delta, \Delta'}\ *r,$$

With our result about excessive theories we can immediately see that a set of bivaluations $\mathbb{B}$ defined by the following conditions:

$\beta \in \mathbb{B}$ if and only if:

- If $\beta(a)=1$ and $\beta(b)=0$ then $\beta(a*b)=0$,
- If $\beta(a)=1$ and $\beta(b)=1$ then $\beta(a*b)=1$,

is a complete semantic for *.

That means we can automatically translate sequent rules into conditions for a set of bivaluations complete for the logic induced by this set of rules (and the converse also holds: we can axiomatize automatically with sequent rules conditions for a set of bivaluations, given in a specific normal form).

Intuitively our result shows that it is sufficient to put $\beta(a)=0$ if $a$ is on the left hand of a sequent, and to put $\beta(a)=1$ if $a$ is on the right. Furthermore, if $a$ and $b$ occur in two different premises of a rules, like in the example, we put "and", but if it is on the same sequent we put "or".

We can conclude saying that the concept of excessivity appears as a key concept in Universal Logic, and using the Bourbakian dialect that it is an essential part of a new "boumologie", a logical "boumologie".

**Notes**

[1]It would be mistaken to think that Universal Logic is a Paramount Logic.

[2]We must recall that Frege himself did not create what we call nowadays first-order logic. Frege is taken as the originator of first-order logic because he has developed the theory of quantification.

[3]According to R.Suszko (see his Introduction to the vol.102 of *Dissertationes Mathematicae*), Lindenbaum was the first to consider the language of logic as an algebra: as an absolute free algebra, and according to Tarski (see his book *Cylindric Algebra*, p.85, n.4), Tarski is the inventor of the so-called Lindenbaum-Tarski algebras. There are two levels in the "realgebrisation" of logic by the Polish logicians, which must not be identified, one due to Lindenbaum, the other one due to Tarski. The terminology Lindenbaum-Tarski algebra is however correct if we think that the idea of Lindenbaum is the first step leading to the Tarskian algebraisation. And this must not reduce the role of Lindenbaum in the development of logic which is not very well-known and which is probably much more important that we can imagine, both on conceptual and technical sides (see the papers of S.J.Surma).

[4]Some people will say that Gentzen's sequent claculus is a counter-example. But in fact Gentzen's systems are systems of deduction which induce logics, and generally they are not taken as generating multi-conclusion logics. And, contrary to some interpretations, Genzten's rules are not conditions on a multi-conclusion relation of deducibility (see our paper "On the distinctions and confusions between, rules and laws, proof and consequence, Gentzen and Tarski").

[5]As it is known this theorem is equivalent to the axiom of choice, this illustrates our remark in §22: we have here a version of the axiom of choice in a "typical" logical framework.

[6]For a more detail account, see [da Costa/Béziau 1993]. Asser used the terminology "vollstandig in bezug auf" which has been translated by "relatively maximal". We don't use it because it is cumbersome. The Brazilians used the word "saturated", but this word is already used in model theory so we thought it was necessary to introduce a new word.

[7]R. Sylvan told the author that in relevant logic the distinction is also fundamental.



[8]This is the version required for completeness of the semantic of maximal theories, the version saying, *every limited theory can be extended in a maximal theory*, is in general not equivalent to it, and in this case doesn't guarantee completeness (see our paper `Excessive Theories').

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