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The Mathematical Structure of Logical Syntax

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ABSTRACT. We analyse the various ways of constructing the set of zero-order formulas (i.e. propositional or sentential formulas) : from the intuitive definition, based on a simple linguistic notion (combination of signs), to the abstract definition of absolutely free algebra. Connecting this last concept with Peano arithmetic, we show why the set of zero-order formulas cannot be axiomatized in first-order logic and explain how this can be used against the formalist approach to logic and mathematics. Finally we try to investigate the historical development of the conception of the set of zero-order formulas.

1. Introduction

A first course of logic usually begins with what is viewed as the most elementary part of logic, namely classical propositional logic. And such a course begins with the definition of the set of formulas.

Most people, even philosophers without any mathematical background, can understand this definition, mainly because it is based on a linguistic intuition : formulas are presented as well-formed expressions built over an alphabet, and the set of formulas is commonly called the "language", and considered as part of "logical syntax".

However there is a very big gap between this intuitive definition and the mathematical concept which is behind it, the concept of absolutely free algebra. In fact, though this concept is used by Polish logicans since at least 40 years (cf. [**LOS**]), most logicians (except those familiar with Polish logic or working in algebraic logic) outside Poland don't know it and don't use it.

Such notion is, for example, never mentioned by philosophers of logic. They still work only with the intuitive linguistic approach in particular when they deal with the ontological problem "sentence versus proposition". Here, to avoid ontological commitment, we will used the neutral Polish terminology "zero-order formulas" and "zero-order logic". [1]

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[1]This terminology is also used by N.M.Martin. He justifies his choice as follows: "it is worth remarking that much of the customary terminology of modern logic is strongly affected by the linguistic views of some of the most prominent contributors to the field, such as Frege, Russell and Carnap. In this book, we will not try either to defend these views or subject them to thorough criticism. (...) Since the terminology commonly used is influenced by these linguistic views, we

Nevertheless we will try to explain the philosophical import of a more sophisticated view on the set of zero-order formulas. We will show that the set of zero-order formulas cannot be axiomatized in first-order logic and explain how this result may dismiss the formalist approach to logic and mathematics.

The fact that such a result is never discussed is symptomatic of the present state of philosophy of logic. On the one hand philosophically oriented logicians ignore it, due to a lack of knowlededge, and continue to speak as if nothing had happened since the thirties ; on the other hand mathematically oriented logicians do not draw the philosophical consequences of such a result and continue to defend a soft formalist viewpoint, which they use, erroneously as we will see here, in order to get rid of all philosophical problems. As Hodges and Kneebone point out: "Many contemporary logicians adopt a formalist view because they do not want to be bothered with philosophical questions" ([**HOK**], p.80).

We will present and analyse three main ways of defining the set of zero-order formulas of zero-order classical logic. We mention zero-order classical logic here in order to fix the ideas, but this specification is not essentially relevant due to the fact that the definition of the set of zero-order formulas is quite the same for all zero-order logics.

We start with the most intuitive definition based on linguistic intuition, we go on with an intermediate one based on concatenation of sequences and we end with the one based on the concept of absolutely free algebra. Then we compare the definition of the set of zero-order formulas with the axiomatization of the set of natural numbers.

In the literature most definitions are a mixture of different levels of abstraction. Our aim here is to distinguish clearly these different levels, and in particular the abstract from the intuitive.

At the end we will make some historical remarks, trying to start an investigation on the historical evolution of the conception of the set of zero-order formulas.

2. Linguistic definition

A very simple way to present the construction of the set of zero-order formulas is to follow the linguistic intuition of alphabetic languages (i.e. language like Greek, English, etc.).

This presentation requires nearly no mathematical background and therefore is useful for philosophers or other people with very little mathematical knowledge or none. We will try to see here how this construction can be presented and understood at this level, and at which point this intuition is idealized.

In alphabetic languages, compound expressions, like words and sentences, are constructed by combinations of signs of an alphabet. Following the same idea we will define a set of logical expressions.

The alphabet A here is formed by three different kinds of signs:

- an infinite quantity of *letters of proposition*, denoted by p_1, p_2, p_3, ...
- signs of punctuation, namely right and left parentheses: (,).

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frequently have to choose between inventing new terms with the risk of being incomprehensible, or else using the customary terms with the risk of, at least, being misleading, and, at worst, reinforcing the Fregean-Logical Positivist family of views (by, so to speak, 'brainwashing'). (...) Faced with this choice, we have, as the reader will see, temporized. (...) we have chooose to use the term (...) 'zero-order' instead of 'propositional' or 'sentential'." ([**MAR**], p.3)

- logical signs : $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\neg$ (negation).

In English, punctation marks are usually not taken as part of the alphabet but are considered as additional signs. Here the notion of alphabet is slightly generalized: an alphabet is a set of signs, which can be classified under various categories. A bigger step from the alphabetic intuition is the idea to consider an infinite set of signs. But the notation used permits an understanding for any person who knows the set of natural numbers (i.e. any literate person).

Logicians like to talk about symbols instead of signs. However most of the time they use the word "symbol" as a synonym of the word "sign". It is good for "épater les bourgeois" and rather confusing for the neophyte who gets impressed by the pompous expression "symbolic logic". In linguistics, the difference between signs and symbols is that in the case of symbols there is a connection with the meaning, by opposition to signs which are blind. In fact logicians do use real symbols, as the symbol $\forall$ introduced by Gentzen. But the formalist approach, based on the rejection of the meaning, does not recognize this true dimension of the symbolism.

A *word*, or an *expression* over the alphabet is any *combination* or *assemblage* of signs of the alphabet, following the standard notion of combination of signs of alphabetic language : string of signs. The concept of combination is understood here in a quite general way, close to the idea of putting together, of physical manipulation.

Here again the usual alphabetic intuition is slightly generalized, because combinations of arbitrary length are considered. In English, though no limitation of size is stipulated, it is easy to see that there is no words of more than, for example, 50 signs.

In English, not all combinations of signs are words. In logic "words" is therefore used in a broader sense. But a "logical language" is also selective : zero-order formulas will be specific words on the alphabet A. In English it is difficult to specify a set of rules of combination of signs which permits to generated all English words and only those. However what linguists have try to do is to determine a set of rules of combination of words which leads to English sentences. This is what is called "generative grammar". But here it is logicians who have influenced linguists and not the contrary. Tarski is the real father of generative grammar and not Chomsky (see the quotation of Corcoran in section 7).

Zero-order formulas are specific combinations of signs, determined by a set of rules, which are called sentences or propositions, from the linguistic perspective. Therefore in logic we go directly from signs to sentences. Sentences are not combinations of words and we recall that what is called word in logic is any combination of signs.

Among the set of all possible combinations of signs over the alphabet A, a particular set is defined, the set of *well-formed expressions* [2] or *sentences*, or *zero-order formulas.*

The definition (DLI) is given by three clauses:

(L1) Starting clause

Letters of propositions are formulas.

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[2]This is the English translation of the French terminology "Expressions bien formées", which is abbreviated by EBFs. In English people use the terminology "Well-formed formulas" abbreviated by WWFs. However this terminology is quite absurd because formulas are generally not considered as any combinations of signs (i.e. words or expressions). Therefore this terminology is pleonastic.

(L2) Generation clause
Given a formula F, $\neg F$ is a formula.
Given two formulas F and G, $(F \wedge G)$, $(F \vee G)$, $(F \rightarrow G)$ are formulas.
(L3) Limitation clause
Nothing else is a formula.

It is not obvious that this definition, as simple as it is, can be understood by an ordinary man, with no mathematical practice. Such an ordinary man may have some difficulty to understand the recursive process involved in (L2) and he will perhaps not see the necessity of (L3) (Historically (L3) was at the beginning not mentioned, see section 7). There is a more intuitive definition (DST) which is a construction by *stages*.

(Stage 0) At this stage, we simply construct formulas by taking all letters of propositions.

(Stage 1) Given any letter of proposition p, we construct the formula $\neg p$. Given any two letters of propositions p and q, we construct the formulas $(p \wedge q)$, $(p \vee q)$, $(p \rightarrow q)$.

(Stage 2) Given a formula F constructed at stage 0 or 1, we construct the formula $\neg F$. Given two formulas F and G constructed at stage 0 or 1, we construct the formulas $(F \wedge G)$, $(F \vee G)$, $(F \rightarrow G)$.

... etc.

A formula is then defined as a combination of signs constructed at one stage. The set of formulas is the union of all formulas constructed at any stage.

Following Enderton's terminology (cf. [**END**], we can say that the first definition (DLI) is from the *top down* and the second (DST) from the *bottom up*. To see that these two defintions are equivalent is not at all obvious without entering into some mathematical considerations.

These definitions are a strange mixture of mathematical and non-mathematical material. They is already a strong idealization of informal linguistic intuition. Notions such as infinite, recursion, set, union, intersection, etc., appear. Of course these notions can be understood at the intuitive level, but very soon we need more precise mathematical concepts. Even for example if we want to prove a simple result saying that any formula has the same number of right and left parentheses. Someone with no idea of what is mathematical induction would not be able to rigorously prove this property.

3. Definition with concatenation of sequences

To the intuitive notion of finite string of signs corresponds the abstract mathematical notion of finite sequence of objects.

A finite sequence of objects of a given set A can be defined within set-theory as a set of pairs, each pair having as a first element a natural number, different from zero, and as a second element an element of A (i.e. as a function from the set of natural numbers to A).

For example the formula $p_4 \wedge p_1$ is represented mathematically by the sequence:

$$\{\langle 3; p_1\rangle, \langle 1; p_4\rangle, \langle 2; \wedge\rangle\}.$$

This is just a way of denoting the mathematical construct. Usually we can denote a sequence, by "abus de language", without mentioning natural numbers, brackets, etc. The formula $p_4 \wedge p_1$ is therefore used as an abbreviation. The order in which we write the signs is enough to express the idea of sequence.

But we have to keep in mind that even if, when we write formulas on paper, this material representation is close to the corresponding mathematical sequence, this does not mean that we are working with this material representation. It is exactly the same case as with geometry where a circle is not a trace on a sheet of paper. We operate a definitive *jump into abstraction.* This means that, for example, we are not taking in account the quality of the ink, the temperature of the air, etc. The mathematical idea of sequence abstracts from the material string only the notion of order, mathematically characterized with the help of natural numbers (note that the property of natural numbers which is used in the definition of sequence is just the fact that they form a discrete order with first element).

Given two sequences $\sigma = \{\langle 1; \rangle, \langle 2; \rangle, ..., \langle n; \rangle\}$ and $\theta = \{\langle 1; \rangle, \langle 2; \rangle, ..., \langle m; \rangle\}$ the concatenation of σ and θ is the sequence $\sigma \star \theta = \{\langle 1; \rangle, \langle 2; \rangle, ..., \langle n; \rangle, \langle 1+n; \rangle, \langle 2+n; \rangle, ..., \langle m+n; \rangle\}$.

Concatenation, denoted here by $\star$, is a binary operation over the set of finite sequences of the alphabet. This operation is not commutative, but it is associative. We have here a structure of semi-group and if we introduce the notion of void sequence, a structure of monoid.

With these two notions of sequences and concatenation of sequences, we can defined (DSC) the set of zero-order formulas in the following manner:

(SC1) <u>Starting clause</u>
Sequences of length 1 formed with "letters of propositions" are proto-formulas.
(SC2) <u>Generation clause</u>
Given a proto-formula F, the sequence of length 1 formed by $\neg$ concatenated with F is a proto-formula.
Given two proto-formulas F and G, $(\star F \star \wedge \star G \star)$, $(\star F \star \vee \star G \star)$, $(\star F \star \rightarrow \star G \star)$ are proto-formulas.
(SC3) <u>Limitation clause</u>
The set of formulas is the intersection of all sets constructed with clauses (SC1) and (SC2), i.e. sets of proto-formulas.

This is a definition by induction within set-theory, in particular because here strings of signs are interpreted as sequences of objects, therefore as a set-theoretical construct.

Given this set-theoretical definition, it makes sense to perform definitions and proofs by induction over the set of formulas. We can also define the set of formulas in terms of sequences and concatenation from the bottom up (stage definition) and show that it is equivalent to the above top down definition.

<u>Definition by induction of substitution</u>
An atomic substitution is a function from the set of atomic formulas (i.e. "letters of propositions") to the set of formulas. This function is extended to a function *sub* from the set of formulas into itself in the following way:

$sub\neg F = \neg \star subF$
$sub(F \wedge G) = (\star subF \star \wedge \star subG \star)$
$sub(F \vee G) = (\star subF \star \vee \star subG \star)$
$sub(F \rightarrow G) = (\star subF \star \rightarrow \star subG \star)$

The fact that an atomic substitution can be properly extended to such a function, and therefore that this definition is correct, is due to a fundamental property which is expressed by the following theorem.

Unique readability theorem

If parentheses were not used in the definition (DSC), then the two formulas $((F \wedge G) \wedge H)$ and $(F \wedge (G \wedge H))$ will be the same formula $F \wedge G \wedge H$, i.e. one and the same sequence. Lukasiewicz has introduced a more "economical" (economical from the point of view of the ink but not from the point of view of readability) method which permits to distinguish these two formulas without using parentheses. This is standardly called "Polish notation". Whitehead and Russell were using also a "notation" without parentheses but with dots (due to Peano).

For all these "notations" it is possible to prove that "there is only one way to write the the same formula". This confuse formulation of the confusely called "unique readability theorem" can be properly understood only in the light of algebraic tools.

In fact, we will see in the next section, that behind this question of "notation" is hidden a mathematical property which is the real essence of the architecture of the set of zero-order formulas.

4. Algebraic and categorical definitions

The set of zero-order formulas defined by (DSC) has a fundamental property which can be roughly described as follows :

If we associate values to atomic formulas, and we have a process which tells us what is the value of a compound formula when the values of its direct components are known, then we can associate values to all formulas.

This property corresponds to the concept of absolutely free algebra.

It is fundamental when extending distributions of truth-values to atomic formulas to the whole set of formulas and also in the defintion of substitution.

The "unique readability theorem" consists just in proving that the set of zero-order formulas constructed with (DSC) is an absolutely free algebra. It turns out that it is nothing more than that : accidental properties of (DSC) related for example to the concept of sequence can be forgotten without any trouble. It is also possible to prove that the set of zero-order formulas constructed with Polish notation or Peano's dots notation are similar free algebras. These three specific representations of the same absolutely free algebra differ only in accidental features of no relevant mathematical siginification (for example formulas in Polish notation are clearly shorter), that is why they are usually taken as equivalent.

Let us now turn to the precise definition of absolutely free algebra. An *absolutely free algebra* A is an abstract algebra (in the sense of Birkhoff, i.e. a set with a family of finitary operations), such that there is a subset G, the set of generators, of the domain $\mathbb{A}$, such that any function from this set to the domain $\mathbb{B}$ of a similar algebra B uniquely extends to a homomorphism from $\mathbb{A}$ to $\mathbb{B}$.

The set of zero-order formulas is an absolutely free algebra generated by atomic formulas with the connectives considered as functions :

$$F = \langle \mathbb{F}; \neg, \rightarrow, \wedge, \vee, \rangle.$$

Endomorphims of this algebra correspond exactly to the standard concept of substitution, as noticed in [**LOS**]. This only one example of nice simplifications that arise when we are working directly with this concept. [3]

It is worth noting that the concept of absolutely free algebra can be defined very elegantly in terms of Category Theory: an absolutre free algebra is the initial object of the category of all abstract algebras of similar type. In fact one proves that for any category of all abstract algebras of similar type, there exists a unique initial object. The fact that an absolutely free algebra is an initial object means that it generates by homomorphic projection all the algebras of the category (see the description of Suszko in section 7).

From this point of view formulas are abstract objects of a mathematical structure. According to the mathematical ontology of the structure, especially developed by Bourbaki, *to be is to be an element of a structure.* Mathematical objects are determined by the relations they have within a structure. Therefore to specify the "nature" of zero-order formulas is to specify the nature of their structure, i.e. the structure of absolutely free algebra.

The question to know which kind of things these abstract objects represent, sentences or propositions, is another problem (see section 6 below).

5. Definition of zero-order formulas and axiomatization of arithmetic

Consider an absolutely free algebra with only one generator, 0, and only one unary function s, called successor. This is what is called a Peano algebra (cf. e.g. [**GRA**]). As the name indicates we are close to natural numbers and Peano axiomatics.

A Peano algebra is the unique model, up to isomorphism, of second order axioms for arithmetic. That it is to say it is the set of natural numbers.

Let us recall second order axioms for arithmetics (2PA). In the language we need just a constant, 0, and a unary function symbol, s.

(2PA1) $\forall x(sx \neq 0)$
(2PA2) $\forall x \forall y(sx = sy \Rightarrow x = y)$
(2PAI) $\forall \varphi((\varphi 0 \,\&\, \forall x \varphi x \Rightarrow \varphi sx) \Rightarrow \forall x \varphi x)$

For obvious reasons we are using here different signs for implication and conjunction. The symbol φ is a variable ranging over monadic predicates.

(2PA) is an axiomatization of the simplest absolutely free algebra. We can in a similar way axiomatize the absolutely free algebra of zero-order formulas. First, note that if we interpret the function s as the negation $\neg$ and 0 as an atomic formula, then the above axioms characterize a zero-order set of formulas with only one atomic formula and negations of this formula. To axiomatize the whole zero-order "language" we need a language with a monadic predicate P (which will help to speak about atomic formulas), a unary function symbol $\neg$ and three binary function symbols, $\rightarrow$, $\wedge$,$\vee$. Here is the axiomatic (2FPA) :

(2PFA1)$\forall x \forall y \forall z((Pz \Rightarrow \neg x \neq z) \,\&\, (x \rightarrow y \neq z) \,\&\, (x \wedge y \neq z) \,\&\, (x \vee y \neq z))$

[3] Polish logicians generally when they are dealing with zero-order logic work directly with the concept of absolutely free algebra (and other concepts of Universal Algebra) without entering in details. Details can be found in [**BAM**].

(2PFA2) $\forall x \forall y((\neg x = \neg y \Rightarrow x = y) \,\&\, (x \to y = u \to t \Rightarrow x = u \,\&\, y = t) \,\&\, (x \wedge y = u \wedge t \Rightarrow x = u \,\&\, y = t) \,\&\, (x \vee y = u \vee t \Rightarrow x = u \,\&\, y = t))$

(2PFAI) $\forall \varphi(((\forall x((Px \Rightarrow \varphi x) \,\&\, (\varphi x \Rightarrow \varphi \neg x)) \,\&\, (\forall x \forall y((\varphi x \,\&\, \varphi y \Rightarrow \varphi(x \to y))) \,\&\, (\varphi x \,\&\, \varphi y \Rightarrow \varphi(x \wedge y)) \,\&\, (\varphi x \,\&\, \varphi y \Rightarrow \varphi(x \vee y)))) \Rightarrow \forall x \varphi x)$

If we replace (2PAI) by a schema of axiom of first-order logic, we get a first-order set of axioms (1PAs). As it is known, it not anymore possible to define addition and multiplication with (1PAs). This already suggests that the models of (1PAs) are not the same as the models of (2PA) and that in particular (1PAs) does not axiomatize the notion of Peano algebra.

(1PAs) is categorical for any uncountable cardinal and therefore is complete and decidable, however it is not $\aleph_0$ categorical. In all uncoutable models, there are some "non-standard" objects, that is to say objects which cannot be reached by a finite number of application of the successor function. Some denumerable models can also have such non-standard objects (that is why this theory is not $\aleph_0$ categorical (see [**END**] and [**VIA**]).

That means that if we interpret the function s as the negation, models of (1PAs) will have some "non-standard formulas" and therefore (1PAs) will not provide a right definition of the set of zero-order formulas constructed only with negation and one atomic formula. Of course the situation will not improve if we consider the first-order formulation (1PFA) of (2FPA) by replacing (2PFAI) by a first-order schema of axiom.

As (1PAs) is complete, it it not possible to find a first-order extension of it which will be $\aleph_0$ categorical, because an extension of (1PAs) is either logically equivalent to it or trivial.

The conclusion is therefore that *it is not possible to find a first-order axiomatization of the set of zero-order formulas.* And this result applies a fortiori to the set of first-order formulas. [4]

This does not contradict the fact that these sets can be constructed within first-order set theory. But this fact does not dismiss the fundamental philosophical import of the above negative result, in particular because first-order set-theory is undecidable and incomplete.

6. Remarks on the formalist approach

It seems that this result can be used against the formalist approach to logic and mathematics.

The formalists think that all mathematics and logic can be founded on formal languages and rules whose manipulation is so simple that it can be operated by any literate person.

This is what Bourbaki argues in order to defend himself against the vicious circle according to which the construction of formal languages requires what it is supposed to found, for example natural numbers :

> Nous ne discuterons pas de la possibilité d'enseigner les principes du langage formalisé à des êtres dont le développement intellectuel n'irait pas jusqu'à savoir lire, écrire et compter. ([**BOU**], pp.4-5)

[4] One referee argues that the above theory which is complete but not categorical "provides full access to the properties of zero-order formulas". But this is ambiguous : it provides full access to the properties of all the objects, including, non-standard formulas. So any theorem of the theory is about standard and non-standard formulas.

But our previous discussion shows very well that to construct the simplest formal language we need much more than the knowledge of primary school. We really need some concepts which can properly be described only by an advanced formal system such as set theory.

Formalists argue also that an essential advantage of their position is one of the fundamental feature of formalism according to which they don't take into account the notion of meaning :

> A theory, a rule, a definition, or the like is to be called formal when no reference is made in it either to the meaning of the symbols (for example, words) or to the sense of the expressions (e.g. sentences), but simply and solely to the kinds and order of the symbols from which the expression are constructed. ([**CAR**], p.1)

We have seen that the construction of the simplest formalized language requires far more that the intuitive notion of order. Moreover the idea of the construction is not reducible to the ability of manipulating signs, and to understand it one needs to understand the "meaning" of a notion such as mathematical induction.

It is also in the spirit of formalism to say that zero-order logic is about sentences rather than propositions.

But, as we have seen, zero-order formulas are mathematical objects which are far more abstract than sentences of natural languages. Given these abstract mathematical objects, we can think that they represent sentences or propositions.

To think that zero-order formulas represent propositions rather than sentences does not necessarily mean that one "believes in proposition". It just means that zero-formulas are abstract objects intended to represent the intuitive notion of proposition as it appears in ordinary mathematics. One may prefer to say that they represent the intuitive notion of sentence, arguing that the intuitive notion of proposition is confused. But the intuitive notion of sentence is not so clear, for example the principle of identity for sentences may appear as problematic (see [**BEZ**]).

What is incorrect, in both cases, is to say that zero-order formulas *are* sentences or that they *are* propositions. But this shall not prohibit us to call them sentences and propositions, using the name of the intuitive notion they are intended to represent. This is a very common and useful mathematical practice, as noticed by H.B.Curry, one of the greatest formalists, who was not afraid to use the term "proposition":

> The term "proposition" is a controversial subject in mathematical logic. Some logicians eschew it like poison ; they insist on replacing it, in all contexts where it had been used as a matter of course, by the word "sentence" ; others insist on using it, ostensibly on the ground that we need to postulate entities which it can properly denote. The usage here is neutral in regard to this metaphysical controversy. (...) a particular kind of interpretation is intended (..) but no commitment as to the metaphysical nature of that interpretation is made. (...) This agrees with the tradition according to which we use terms which suggest the intended application .(...) Mathematicians can and do use, as technical terms, words which are commonly used for other purposes in unrelated context. The use of the term "proposition" in

> a mathematical context does not commit one, unless he so chooses, to postulate mysterious entities of an esoteric sort. ([**CUR**], pp.168-169).

7. Historical remarks

It seems difficult to trace back the definition of zero-order formulas from the jungle of logical works written before logic turns to be officialized and standardized in classical text books such as those of Church and Kleene.

According to Corcoran, Tarksi is the real creator of the precise definition of set of formulas (close to the definition (DSC) that we have presented):

> In order to treat the syntax of the object language within a deductive metasystem it is necessary to present a formal definition of the set of sentences of the object language. Such a definiton has become known as a recursive grammar, and, more recently, as a generative grammar. In order to present a formally correct generative grammar it is necessary to conceive of string theory (the laws governing the interrelations among strings of signs) and to codify string theory as a deductive science. In the "Wahrheitsbegriff", Tarski isolates as a primitive notion the fundamental operation of concatenation of strings and he presents, employing concatenation, the first axiomatic codification of string theory (VIII, 173), thereby providing deductive foundations of scientific syntax. Article VIII also contains the first formal presentation of a generative grammar (VIII, 175, 176). It is to be regretted that many linguists, philosophers, and mathematicians know so little of the history of the methodology of deductive science that they attribute the basic ideas of generative grammar to linguists working in the 1950s rather than to Tarski (and other logician/methodologists) working in the early 1930s. Other authors including Leśniewski, Ajdukiewicz, Lukasiewicz, Thue, Post, and Gödel had used ideas about scientific syntax before 1932. But these authors presupposed in an informal way the ideas that Tarski presented in a careful deductive setting. This is one more instance where Tarski provides the formal foundations of ideas already informally in use. ([**COR**], p.xx) [5]

We will provide a few quotations from some of the most famous texts and papers of logic of the beginning of the century which seem to support Corcoran's statement.

In *Principia Mathematica*, we find the following presentation:

> An aggregation of propositions, considered as wholes not necessarily unambiguously determined, into a single proposition more complex than its constituents, is a function with propositions as arguments. The general idea of such an aggregation of propositions, or of variables representing propositions, will not be employed in this work. But there are four special cases which are of fundamental importance, since all the aggregations of subordinate propositions into one complex proposition which occur in the sequel are formed out of them step by step. They are (1) the Contradictory function, (2) the Logical

[5]More information can be found in [**COA**].

> Sum, or Disjunctive Function, (3) the Logical Product, or Conjunctive function, (4) the Implicative Function. (...) The Logical Product is a propositional function with two arguments p and q, and is the proposition asserting p and q conjunctively, that is, asserting that both p and q are true. ([**WHR**], p.6)

This quotation shows clearly that the definition of the set of formulas was quite intuitive and in particular that syntactic and semantical considerations were mixed.

Many years later, the same occurs in the textbook of Hilbert and Ackermann:

> Aussagen können in bestimmter Weise zu neuen Aussagen verknüpft werden. Z.B. kann man aus den beiden Aussagen "2 is kleiner als 3", "der Schnee is schwarz" (sic) die neuen Aussagen bilden : "2 ist kleiner als 3 *und* der Schnee ist schwarz" (...). $X\&Y$ (lies "X *und* Y") bezeichnet die Aussage, die dann und nur dann richtig ist, wenn sowohl X als Y richtig is. ([**HIA**], p.3).

Post and Gentzen who respectively worked on the influence of Whitehead-Russell and Hilbert-Ackermann's books, gave both a purely formal syntactic definition of the set of zero-order formulas. But their definitions are still far from Tarski's achievement.

Here is Post's definition (note that what Post calls "elementary propositions" are not atomic propositions but propositions of zero-order logic by opposition of higher-order propositions):

> Let $p, p_1, p_2, ..., q, q_1, q_2, ..., r, r_1, r_2, ...$ arbitrarily represent the variable elementary propositions mentionned in the introduction.Then by means of the two primitive functions $\sim p$ (read not p, the function of negation) and $p \vee q$ (p or q, the function of disjunction) with the aid of the primitive propositions
>
> I. If p is an elementary propositions, $\sim p$ is an elementary proposition,
>
> II. If p and q are elementary propositions, $p \vee q$ is an elementary proposition,
>
> we combine these variables to form the various propositions, or, rather, ambiguous values of propositional functions of the system. ([**POS**], p.164).

The definition of Post bears several defects. No reference to the starting clause is made and there is no limitation clause. Post, like Whitehead-Russell interprets the connectives as functions (due to their intended semantical interpretation) but we are still far from the notion of absolutely free algebra.

Gentzen, in his famous paper, constructs the set of first-order formulas without any reference to meaning, stating clearly the starting and the generative clause but without mention of the limitation clause (cf. [**GEN**], items 2.1, 2.11, 2.12). However he explicitly says that the concept of formula is defined inductively (item 2.11).

During all this pre-Tarskian period, it is not at all clear what is the flesh and blood of formulas. They are probably considered as linguistic objects, but at the same time start to be treated as mathematical objects.

It is difficult to know exactly when it was realized that the set of zero-order formulas can be characterized by the concept of absolutely free algebra, although for sure it is an achievement of the Polish school. In one paper, Suszko sustains that this idea goes back to Lindenbaum:

> ... as observed first by Lindembaum, formalized languages are algebraic systems, i.e., sets supplied with (free) operations determined by formation rules. ([**SUa**], p.5)

However, in another paper Suszko says that it is an idea of Lindenbaum and Tarski:

> Lindenbaum and Tarski observed that the formalized language L is an absolutely free, or anarchic algebraic structure and, hence, the fountain of the whole class K(L) of all algebraic structures similar to L ([**SUb**], p.377). [6]

Now let us turn to the relation between Peano arithmetic and the construction of the set of zero-order formulas.We will examine here a text by Henkin where he also claims the priority of Tarksi's contribution:

> Tarski a été le premier à développer de façon axiomatique la théorie des expressions ([**HEN**], p.23).

Henkin presents three axioms for the construction of expressions (independent of any particular alphabet) and an operation of concatenation $\cap$ ("enchainement") between expressions (considered implicitly as sequences). He says that the set of expressions together with the operation of concatenation form a free semi-goup and makes the following comments:

> Observons que dans le cas $n = 1$, les axiomes E1-E3 ne sont autres que les axiomes de Peano pour le système des nombres naturels ; l'opération $\cap$ est alors l'addition des nombres naturels. Nous sommes donc portés à croire que la théorie des expressions n'est pas moins riche, dans son développement, que la théorie des nombres. En fait il n'en est rien . ([**HEN**], 1956, pp.22-23)

It seems that here Henkin does not make the right comparison between the set of formulas and arithmetic. In particular because, as we have seen, the notion of concatenation is not essential for zero-order formulas. But what he says can be transposed to the absolutely free algebra of zero-order formulas.

Henkin argues further on that the theory of expressions is simpler than number theory because it is decidable. For sure, it is simpler, but we have seen that though the theory (1PAF) is decidable, this theory does not axiomatize properly the set of zero-order formulas. To define the set of zero-order formulas we need a theory which is not so simple (like first-order set-theory).

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[6] It is sometimes difficult to distinguish the respective contributions of Tarski and Lindenbaum. They were close collaborators and both brillant minds. Due to the unfortunately premature death of Lindenbaum his titanic work is not well-known. There is a big work here for the historian of logic (there is a Master Thesis about Lindenbaum's work : [**SZU**]) . In fact Lindenbaum's works and ideas turned to be known through Tarski who honestly always recalled the important contributions of Lindenbaum. Tarski also recalled that one of the most famous concept to which the name of Lindenbaum is attached, Lindenbaum algebra, is in fact due not to Lindenbuam but to him.

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