There are no self-extensional three-valued implicative paraconsistent logics

Arnon Avron and Jean-Yves Beziau

Abstract. A proof is presented showing that there is no paraconsistent logic with implication which has a three-valued characteristic matrix and in which the replacement principle holds.

Mathematics Subject Classification (2000). Primary 03B53, Secondary 03B50.

Keywords. Paraconsistent logic, Many-valued logic, Self-extensionality, Negation, Implication.

1. Introduction

A paraconsistent logic is a logic with a paraconsistent negation. A paraconsistent negation is defined on the basis of the rejection of the law of explosion. This means that there are propositions φ and ψ such that:

$$\varphi, \neg\varphi \nvdash \psi$$

This definition by itself is not satisfactory. Some other properties have to hold unless we allow any unary connective to be considered as a *negation*. Unfortunately, it is not universally agreed yet which cluster of properties and metaproperties should be considered to guarantee that we are dealing with a negation.[1] The general idea is to have a good part of the properties of classical negation which are compatible with the rejection of explosion on the one hand, and perhaps natural stronger forms of that principle on the other hand. This includes first of all negative properties of negation. Very natural examples of such properties are $p \nvdash \neg p$ when p is an atomic formula, and $p, \neg p \nvdash \neg q$ when p and q are distinct atomic formulas.[2] However, *positive* properties should be considered as well. Examples here may be double-negation rules, and rules for interaction of negation with other connectives, such as $\neg(\varphi \wedge \psi) \vdash \neg\varphi \vee \neg\psi$.

[1]See [6, 8, 9, 13, 1, 2, 3] for some studies of this problem.

[2]The later property has been called called *strong paraconsistency* in [4]. It is a special case of *strict paraconsistency* ([17]), as well as *bold paraconsistency* ([10]).

An important positive property at the level of metaproperties is the *replacement property* (not to be confused with the *substitution property*). A logic has this property if it allows to *replace* an occurrence of a formula by a logically equivalent one, where two formulas φ and ψ are called equivalent in a logic (notation: $\varphi \dashv\vdash \psi$) if each of them can be deduced in that logic from the other. For example, suppose a logic has the replacement property, p is logically equivalent in it to $p \wedge p$, and $p \wedge q \vdash p$. Then $p \wedge q \vdash p \wedge p$ as well (the second occurrence of p having been replaced by $p \wedge p$). Following a terminology introduced by Polish logicians, a logic which has the replacement property is called *self-extensional.*

The replacement property is an important metaproperty valid in classical logic, intuitionistic logic, all intermediate logics, all normal modal logics, and many more. An interesting question concerning paraconsistent logics is to what extent this metaproperty is compatible with the notion of paraconsistent negation. Concerning this, it is known that many paraconsistent logics do *not* have this property. Thus in the paraconsistent logic **C1** of Newton da Costa [11] we have $p \supset p \dashv\vdash q \supset q$ but we do not have $\neg(p \supset p) \dashv\vdash \neg(q \supset q)$. What is more, Urbas has shown that there are no paraconsistent extensions of **C1** in which the replacement theorem holds (cf. [16]).

Another negative result concerning self-extensional paraconsistent logics was given in [7], where it is shown that paraconsistent logics in which both $\varphi \dashv\vdash \neg\neg\varphi$ and $\vdash \neg(\varphi \wedge \neg\varphi)$ hold cannot be self-extensional. A consequence of this result is that basic three-valued logics like Asenjo's logic [5] (renamed **LP** by Priest [14]) and da Costa and D'Ottaviano's logic **J3** ([12]) are not self-extensional. The same is known to be true for another famous three-valued paraconsistent logic: Sette's logic P1 [15], even though in this logic we do not have $\vdash \neg(\varphi \wedge \neg\varphi)$. These examples raise the question whether *every* three-valued paraconsistent logic (with a reasonable expressive power) necessarily lacks the replacement property.

In the present paper we provide a partial answer to the above question. It is shown that there are no *implicative* paraconsistent self-extensional three-valued logics. By "implicative" we mean a logic which possesses an implication (that is, a connective $\supset$ such that $\mathcal{T}, \varphi \vdash \psi$ iff $\mathcal{T} \vdash \varphi \supset \psi$). This negative result is important, because three-valued matrices provide the most basic tool for the development of non-classical logics in general, and paraconsistent logics in particular.

2. Basic Definitions

In what follows we denote by $\mathcal{L}$ a propositional language with a set $\mathsf{Atoms}(\mathcal{L}) = \{P_1, P_2, \ldots\}$ of atomic formulas, and use p, q, r to vary over this set. The set of the well-formed formulas of $\mathcal{L}$ is denoted by $\mathcal{W}(\mathcal{L})$ and $\varphi, \psi, \phi, \sigma$ will vary over its elements. The set $\mathsf{Atoms}(\varphi)$ denotes the atomic formulas occurring in φ.

For the reader's convenience, we now review some relevant definitions.

Definition 2.1. A *logic* is a pair $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$, such that $\mathcal{L}$ is a language, and $\vdash$ is a structural and non-trivial Tarskian consequence relation for $\mathcal{L}$.

Definition 2.2. Let $\mathbf{L} = \langle \mathcal{L}, \vdash_\mathbf{L} \rangle$ be a logic.

- Formulas $\psi, \varphi \in \mathcal{W}(\mathcal{L})$ are *equivalent* in $\mathbf{L}$, denoted by $\psi \dashv\vdash_\mathbf{L} \varphi$, if $\psi \vdash_\mathbf{L} \varphi$ and $\varphi \vdash_\mathbf{L} \psi$.
- Formulas $\psi, \varphi \in \mathcal{W}(\mathcal{L})$ are *congruent* (or *indistinguishable*) in $\mathbf{L}$, denoted by $\psi \equiv_\mathbf{L} \varphi$, if for every formula σ and atom p it holds that $\sigma[\psi/p] \dashv\vdash_\mathbf{L} \sigma[\varphi/p]$.
- $\mathbf{L}$ has the *replacement property*, or is *self-extensional* ([18]), if any two formulas which are equivalent in $\mathbf{L}$ are congruent in it.

Definition 2.3. Let $\mathbf{L} = \langle \mathcal{L}, \vdash_\mathbf{L} \rangle$ be a propositional logic. A binary connective $\supset$ of $\mathcal{L}$ is called an *implication for* $\mathbf{L}$ if $\supset$ has in $\vdash_\mathbf{L}$ the following property (which both classical and intuitionistic implications have, and which characterizes the latter):

$$\mathcal{T}, \varphi \vdash_\mathbf{L} \psi \text{ iff } \mathcal{T} \vdash_\mathbf{L} \varphi \supset \psi.$$

Definition 2.4. A *matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

- $\mathcal{V}$ is a non-empty set of truth values;
- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, called the *designated* elements of $\mathcal{V}$;
- $\mathcal{O}$ is a function that associates an n-ary function $\widetilde{\diamond}_\mathcal{M} : \mathcal{V}^n \to \mathcal{V}$ to each n-ary connective $\diamond$ of $\mathcal{L}$.

Definition 2.5. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.

- An $\mathcal{M}$*-valuation* for $\mathcal{L}$ is a function $\nu : \mathcal{W}(\mathcal{L}) \to \mathcal{V}$ such that for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}(\mathcal{L})$, $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}_\mathcal{M}(\nu(\psi_1), \ldots, \nu(\psi_n))$. We denote the set of all the $\mathcal{M}$-valuations by $\Lambda_\mathcal{M}$.
- A valuation $\nu \in \Lambda_\mathcal{M}$ is an $\mathcal{M}$*-model* of a formula ψ, if it belongs to the set $mod_\mathcal{M}(\psi) = \{\nu \in \Lambda_\mathcal{M} \mid \nu(\psi) \in \mathcal{D}\}$. The $\mathcal{M}$-models of a theory $\mathcal{T}$ are the elements of the set $mod_\mathcal{M}(\mathcal{T}) = \cap_{\psi \in \mathcal{T}} mod_\mathcal{M}(\psi)$.

In the sequel, we shall sometimes omit the prefix '$\mathcal{M}$' from the notions above. Also, when it is clear from the context, we shall omit the subscript '$\mathcal{M}$' in $\widetilde{\diamond}_\mathcal{M}$.

Definition 2.6. Given a matrix $\mathcal{M}$, the consequence relation $\vdash_\mathcal{M}$ that is *induced by* (or associated with) $\mathcal{M}$, is defined by: $\mathcal{T} \vdash_\mathcal{M} \psi$ if $mod_\mathcal{M}(\mathcal{T}) \subseteq mod_\mathcal{M}(\psi)$. We denote by $\mathbf{L}_\mathcal{M}$ the pair $\langle \mathcal{L}, \vdash_\mathcal{M} \rangle$, where $\mathcal{M}$ is a matrix for $\mathcal{L}$ and $\vdash_\mathcal{M}$ is the consequence relation induced by $\mathcal{M}$.

3. Paraconsistent Three-valued Matrices

Since this paper aims to show that three-valued paraconsistent logics with certain properties do not exist, we should be very clear about what we mean by the term "paraconsistent logics". A useful general definition can be found in [3]. However, for the purposes of this paper the following much weaker notion would do.

Definition 3.1. Let $\mathbf{L} = \langle \mathcal{L}, \vdash_\mathbf{L} \rangle$ be a propositional logic whose language $\mathcal{L}$ includes the unary connective $\neg$.

1. $\neg$ is called a *weak negation* for $\mathbf{L}$ if the following conditions are satisfied:
    - $p \nvdash_\mathbf{L} \neg p$ if p is atomic.

- $\neg p \nvdash_{\mathbf{L}} p$ if p is atomic.
- There is no formula φ such that both $\vdash_{\mathbf{L}} \varphi$ and $\vdash_{\mathbf{L}} \neg\varphi$.[3]

2. A logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ is *$\neg$-paraconsistent* if $\neg$ is a weak negation for $\mathbf{L}$, and there are atoms p, q such that $p, \neg p \nvdash_{\mathbf{L}} q$.[4]

From now on we assume that $\mathcal{L}$ includes $\neg$, and write just "paraconsistent" instead of "$\neg$-paraconsistent". We also write $\dashv\vdash_{\mathcal{M}}$ instead of $\dashv\vdash_{\mathbf{L}_{\mathcal{M}}}$, $\equiv_{\mathcal{M}}$ instead of $\equiv_{\mathbf{L}_{\mathcal{M}}}$, etc.

Definition 3.2. A matrix $\mathcal{M}$ for $\mathcal{L}$ is *paraconsistent* if $\mathbf{L}_{\mathcal{M}}$ is paraconsistent.

Proposition 3.3. *Every 3-valued paraconsistent matrix is isomorphic to a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ in which $\mathcal{V} = \{t, f, \top\}$, $\mathcal{D} = \{t, \top\}$, $\tilde{\neg} t = f$, $\tilde{\neg} f \in \mathcal{D}$ and $\tilde{\neg} \top \in \mathcal{D}$.*

Proof. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a 3-valued paraconsistent matrix. Since $p \nvdash_{\mathcal{M}} \neg p$, there are two distinct elements t and f in $\mathcal{V}$ such that $t \in \mathcal{D}$, $f \notin \mathcal{D}$, and $\tilde{\neg} t = f$. Since $p, \neg p \nvdash_{\mathbf{L}} q$, necessarily there is also an element $\top$ in $\mathcal{V}$ such that both $\top \in \mathcal{D}$ and $\tilde{\neg} \top \in \mathcal{D}$ (and so $\top$ is distinct from both t and f). Finally, since $\neg p \nvdash_{\mathcal{M}} p$, and f is the only non-designated element of $\mathcal{V}$, necessarily $\tilde{\neg} f \in \mathcal{D}$. $\square$

4. The Main Theorem

In this section we assume, without loss of generality, that every 3-valued paraconsistent matrix has the form described in Proposition 3.3.

Proposition 4.1. *Let $\mathcal{M} = \langle \{t, f, \top\}, \{t, \top\}, \mathcal{O} \rangle$ be a 3-valued paraconsistent matrix. Then $\varphi \equiv_{\mathcal{M}} \psi$ in $\mathcal{M}$ iff $\nu(\varphi) = \nu(\psi)$ for every assignment ν in $\mathcal{M}$.*

Proof. The condition is obviously sufficient. To show that it is also necessary, assume that $\nu(\varphi) \neq \nu(\psi)$ for some assignment ν in $\mathcal{M}$. If $\nu(\varphi) = f$ and $\nu(\psi) \neq f$ then $\psi \nvdash_{\mathbf{L}_{\mathcal{M}}} \varphi$. If $\nu(\varphi) = t$ and $\nu(\psi) \neq t$ then $\neg\psi \nvdash_{\mathcal{M}} \neg\varphi$. Similarly, if $\nu(\psi) = f$ and $\nu(\varphi) \neq f$ then $\varphi \nvdash_{\mathcal{M}} \psi$, while if $\nu(\psi) = t$ and $\nu(\varphi) \neq t$ then $\neg\varphi \nvdash_{\mathcal{M}} \neg\psi$. Hence in all cases $\varphi \not\equiv_{\mathcal{M}} \psi$. $\square$

Theorem 4.2. *Let $\mathcal{M}$ be a 3-valued paraconsistent matrix, and suppose $\mathbf{L}_{\mathcal{M}}$ has an implication $\supset$. Then $\mathbf{L}_{\mathcal{M}}$ is not self-extensional.*

Proof. First observe that the fact that $\supset$ is an implication for $\mathcal{M}$ implies that $\mathcal{T} \vdash_{\mathcal{M}} \varphi$ whenever $\mathcal{T} \vdash_{\mathbf{I}_{\supset}} \varphi$, where $\mathbf{I}_{\supset}$ is the implicational fragment of intuitionistic logic. In what follows we freely use this fact.

Now assume that $\mathbf{L}_{\mathcal{M}}$ does have the replacement property. To reach a contradiction, we need some facts that follow from our assumptions about $\mathcal{M}$.

1. $t \tilde{\supset} f = \top \tilde{\supset} f = f$.
   Proof: Since $p, p \supset q \vdash_{\mathbf{I}_{\supset}} q$, also $p, p \supset q \vdash_{\mathcal{M}} q$. It follows that if $\nu(p) \in \mathcal{D}$ and $\nu(q) = f$ then necessarily $\nu(p \supset q) = f$. Hence $t \tilde{\supset} f = \top \tilde{\supset} f = f$.

[3] This condition was not included in the definition of "weak negation" used in [1]. The first two conditions have been considered in [13].

[4] Since $\vdash_{\mathbf{L}}$ is structural, this implies that $p, \neg p \nvdash_{\mathbf{L}} q$ whenever p and q are distinct atoms.

2. $a\tilde{\supset}a = t$ for every $a \in \{t, \top, f\}$.
   Proof: Since $\supset$ is an implication for $\mathbf{L}_{\mathcal{M}}$, $\vdash_{\mathcal{M}} \varphi \supset \varphi$ for every φ. Hence $a\tilde{\supset}a \in \mathcal{D}$ for every $a \in \{t, \top, f\}$. The fact that $\vdash_{\mathcal{M}} \varphi \supset \varphi$ for every φ implies also that $p \supset p \dashv\vdash_{\mathcal{M}} q \supset q$ for every atomic formulas p and q. Therefore it follows from Proposition 4.1 that $\nu(p \supset p) = \nu(q \supset q)$ for every assignment ν in $\mathcal{M}$, and every p and q. Hence either $a\tilde{\supset}a = t$ for every $a \in \{t, \top, f\}$, or $a\tilde{\supset}a = \top$ for every $a \in \{t, \top, f\}$. Had the latter been the case, we would have got that $\nu(\neg(p \supset p)) \in \mathcal{D}$ for every assignment ν in $\mathcal{M}$, implying that $\neg(p \supset p)$ is a theorem of $\mathbf{L}_{\mathcal{M}}$. Since $p \supset p$ is a theorem of $\mathbf{L}_{\mathcal{M}}$, This would have contradicted the assumption that $\neg$ is a weak negation for $\mathbf{L}_{\mathcal{M}}$. (See the third condition in the definition of weak negation.) It follows that $a\tilde{\supset}a = t$ for every $a \in \{t, \top, f\}$.
3. $f\tilde{\supset}a = t$ for every $a \in \{t, \top, f\}$.
   Proof: The previous item and Proposition 3.3 imply that $\tilde{\neg}(a\tilde{\supset}a) = f$ for every $a \in \{t, \top, f\}$. It follows that $\neg(p \supset p)$ has no model in $\mathcal{M}$, and so $\neg(p \supset p) \vdash_{\mathcal{M}} p$. Since $\supset$ is an implication for $\mathbf{L}_{\mathcal{M}}$, this in turn implies that $\vdash_{\mathcal{M}} \neg(p \supset p) \supset p$, and so $p \supset p \dashv\vdash_{\mathcal{M}} \neg(p \supset p) \supset p$. Since $a\tilde{\supset}a = t$ for every a, and $\tilde{\neg}t = f$, the last fact implies that $t = f\tilde{\supset}a$ for every $a \in \{t, \top, f\}$.
4. $\tilde{\neg}f = t$.
   Proof: Since $\neg p, p \vdash_{\mathcal{M}} \neg p$, and $\supset$ is an implication for $\mathbf{L}_{\mathcal{M}}$, $\neg p \vdash_{\mathcal{M}} p \supset \neg p$. On the other hand $p \supset \neg p \vdash_{\mathcal{M}} \neg p$, since if $\nu(\neg p) = f$ then $\nu(p) = t$, and so $\nu(p \supset \neg p) = f$ by item 1 above. It follows that $\neg p \dashv\vdash_{\mathcal{M}} p \supset \neg p$. Hence Proposition 4.1 implies that $\nu(\neg p) = \nu(p \supset \neg p)$ for every ν and p. In particular, if $\nu(p) = f$ we get that $\tilde{\neg}f = f\tilde{\supset}\tilde{\neg}f = t$ (by item 3).

Now a simple computation using items 2 and 1 (together with Proposition 3.3) shows that if p is atomic then $p \vdash_{\mathcal{M}} \neg(p \supset \neg(p \supset p))$. On the other hand it follows from item 3 (or from item 2) that if $\nu(p) = f$ then $\nu(\neg(p \supset \neg(p \supset p)) = f$, implying that $\neg(p \supset \neg(p \supset p)) \vdash_{\mathcal{M}} p$. Hence $p \dashv\vdash_{\mathcal{M}} \neg(p \supset \neg(p \supset p))$. However, by letting $\nu(p) = \top$ we get (using items 2, 1, and 4) that $\neg p \nvdash_{\mathcal{M}} \neg\neg(p \supset \neg(p \supset p))$. This contradicts the assumed replacement property of $\mathbf{L}_{\mathcal{M}}$. □

Acknowledgments

The first author is supported by The Israel Science Foundation under grant agreement no. 817/15.

This paper was written during a stay of the second author at University of Tel Aviv within the GeTFun exchange program - Marie Curie project PIRSES-GA-2012-318986 funded by EU-FP7.

References

[1] O. Arieli, A. Avron, and A. Zamansky, *Maximal and premaximal paraconsistency in the framework of three-valued semantics*, Studia Logica, **97** (2011), 31–60.

[2] O. Arieli, A. Avron, and A. Zamansky, *Ideal paraconsistent logics*, Studia Logica, **99** (2011), 31–60.

[3] O. Arieli and A. Avron, *Three-valued paraconsistent propositional logics*, In J.-Y. Beziau, M. Chakraborty, and S. Dutta, editors, **New Directions in Paraconsistent Logic**, 91–129, Springer India, 2015.

[4] A. Avron, B. Konikowska, and A. Zamansky, *Cut-free sequent calculi for C-systems with generalized finite-valued semantics*, Logic and Computation, **23** (2013), 517–540.

[5] F.G. Asenjo, *A calculus of antinomies*, Notre Dame Journal of Formal Logic, **7** (1966), 103–105.

[6] J.-Y. Beziau, *Théorie législative de la négation pure*, Logique et Analyse, **147-148** (1994), 209–225.

[7] J.-Y. Beziau, *Idempotent full paraconsistent negations are not algebraizable*, Notre Dame Journal of Formal Logic, **39** (1998), 135–139.

[8] J.-Y. Beziau, *What is paraconsistent logic?*, in **Frontiers of paraconsistent logic**, D.Batens et al. (eds), Research Studies Press, Baldock, 2000, 95–111.

[9] J.-Y. Beziau, *Are paraconsistent negations negations ?*, in **Paraconsistency: the logical way to the inconsistent**, W.Carnielli et al. (eds), Marcel Dekker, New-York, 2002, 465–486.

[10] W.A. Carnielli, M.E. Coniglio, and J. Marcos, *Logics of formal inconsistency*, in **Handbook of Philosophical Logic (2nd edition), vol. 14**, D.M. Gabbay and F. Guenthner (eds), Springer, 2007, 1–95.

[11] N.C.A. da Costa, *On the theory of inconsistent formal systems*, Notre Dame Journal of Formal Logic, **15** (1974), 497–510.

[12] I.M.L. D'Ottaviano and N.C.A. da Costa, *Sur un problème de Jaskowśki*, Cr. R. Acad Sc. Paris, **270** (1970), 1349–1353.

[13] J. Marcos, *On negation: Pure local rules*, Journal of Applied Logic, **3**, (2005), 185–219.

[14] G. Priest, *The logic of paradox*, J. Philos. Logic, **8** (1979), 219–241.

[15] A.M. Sette, *On the propositional calculus P1*, Mathematica Japonica, **16** (1973), 173–180.

[16] I. Urbas, 1987 *On Brazilian paraconsistent logics*, PhD, Australian National University, Canberra.

[17] I. Urbas, *Paraconsistency*, Studies in Soviet Thought, **39** (1989), 343–354.

[18] R. Wójcicki, **Theory of Logical Calculi: Basic Theory of Consequence Operations**, Kluwer Academic Publishers, 1988.

Arnon Avron
School of Computer Science
Tel Aviv University, Israel
e-mail: `aa@cs.tau.ac.il`

Jean-Yves Beziau
UFRJ - Federal University of Rio de Janeiro, RJ, Brazil
CNPq - Brazilian Research Council
e-mail: `jyb@jyb-logic.org`