Aspects of Paraconsistent Logic

NEWTON C. A. DA COSTA, JEAN-YVES BÉZIAU and OTÁVIO A. S. BUENO, *Institute for Advanced Studies, University of São Paulo, Av. Prof. Luciano Gualberto, trav. J, 374, 05508-010 São Paulo, SP, Brazil. E-mail: {ncacosta,beziau,oasbueno}@cat.cce.usp.br*

1 Introduction

During the last two decades, we have been witnessing a growing interest as well as a remarkable activity in connection to paraconsistent logic. So many specialized publications appearing, international congresses being held, and research papers advancing considerably the theory could not but confirm the highly favorable environment conditions for this new kind of logic. More recently, however, in addition to the well-known contributions to the issues linked to the foundations of mathematics, paraconsistent logic has found a fruitful (and, for sure, amazing) new field of application: Computer Science! Indeed, specially within Artificial Intelligence, in order to face theoretical difficulties raised by inconsistent data base, paraconsistent tools have been successfully applied. As a result, new paraconsistent systems have been developed, opening thus new research fields, as well as the way to potentially interesting applications.

Our aim in this rather expository paper is to present, in an outline, some of the main features of a logic of inconsistent but non-trivial systems. There are, in fact, infinitely many paraconsistent logics, and in what follows we shall examine mainly one of them, a specific paraconsistent propositional logic called C_1^+. In order to do so, this work is divided into four sections. In the following, we shall consider, from a historical point of view, some aspects underlying the inception of paraconsistent logic in general, as well as of C_1^+ in particular. Thus (or so we hope) some of the motivations in order to examine this specific paraconsistent system might be supplied. In Section 3, a technical development will then be presented, with a particular emphasis on some results recently obtained regarding C_1^+. After considering this system, in Section 4 some straightforward, possible applications of it are to be then briefly suggested. However, given its character as a non-classical logic, no general exposition of paraconsistency can be minimally acceptable without some philosophical remarks regarding its nature. Hence, in the last section, some brief comments on this issue will be presented, as far as some aspects of C_1^+ are concerned, at least in order not to leave this demand totally unsatisfied.

2 Historical aspects

As it is well known, and solidly established at present, in order to find the main forerunners of paraconsistent ideas, one should trace as far back as Jan Łukasiewicz's and Nicolaj Vasili'év's work. Indeed, in 1910/11, quite independently of each other, both of them have stressed the importance of a revision of some laws of Aristotelian logic, opening up in this way the possibility of the development – in an analogy with non-Euclidean geometry – of non-Aristotelian logics, mainly the ones in which the principle of contradiction is somewhat restricted.

In his celebrated 1910 work, *On the Principle of Contradiction in Aristotle*, as well as in a related paper from the same period, Łukasiewicz, after presenting three different Aristotelian formulations of the principle of contradiction – an ontological, a logical and a psychological one – and rejecting each of them, argues that such a principle could not be so basic as one usually supposes. As a consequence, a precedent was created for the beginning of non-classical logic; however, unable to elaborate a particular logical system at this time, the precedent, to a certain extent, was lost.

Similarly, Vasili'év, although not having formulated himself a specific system, because of his ideas related to imaginary logic, is rightly considered as a precursor of paraconsistent theories. On this regard, it should be noted the deep inspiration extracted from Lobatchevsky's work on non-Euclidean geometry: more than its name (at the time, this geometry was known as imaginary geometry), the methods of construction were also strikingly similar to the ones used by Vasili'év. Furthermore, according to Arruda (see [2]), Vasili'év believed that, just as within Lobatchevski's geometry, his logic could also present a classical interpretation.

But it was not earlier than 1948 that Stanislaw Jaskowski, under Łukasiewicz's influence, would propose the first paraconsistent propositional calculus[1]. So, he was possibly the first to formulate within inconsistent theories the issues connected with non-triviality. Indeed, one of the basic, explicit conditions to be met by his system is that when applied to contradictory theories, it should not be the case that all their formulas become a thesis of the system; that is, the presence of contradictions should by no means entail the system's trivialization.

In close connection to this point, Jaskowski's paraconsistent theories, from Arruda's viewpoint (see [3]), have roughly been developed to fulfill three basic motivations: (1) to provide a conceptual machinery to approach the problem of deductively systematizing theories that contain contradictions, in particular the ones (2) whose contradictions are generated by vagueness, and finally, (3) to study some empirical theories containing contradictory postulates.

Nevertheless, however important Jaskowski's work has been (being, in fact, tremendously relevant), it is to one of the authors of this paper (Newton da Costa) that is generally credited the origins of paraconsistent logic as it is known today. Since 1954, in fact, he has formulated, in an independent way, many such systems, ranging from the propositional to the predicate levels (with or without identity), as well as to some calculi of descriptions and numerous applications to set theory.

The first published work of da Costa presenting a system of paraconsistent logic

[1] It should be mentioned, however, as Kosta Dosen has pointed out (see [24]), that in 1928 I. E. Orlov presented an axiomatization of the implication-negation fragment of the relevant logic R. Nevertheless, despite the fact that such a logic is a paraconsistent one, given that Orlov seems to not to have had any intention of formalizing an inconsistent but non-trivial system, we shall not consider him, as opposed for instance to E. Alves' view (cf. [1]), as a forerunner of paraconsistent logic.

is [13], entitled 'Calculs propositionnels pour les systèmes formels inconsistants' (although his work on the subject had begun several years before). For a long time, da Costa used the expression 'systèmes formels inconsistants' to refer to his systems, until a friend of his, F. Miró Quesada, under his request, proposed in correspondence with him and in a lecture during the Third Latin-American Symposium of Mathematical Logic (held in Campinas, in 1976) the name 'paraconsistent logic'. Many years later (see [16]), da Costa would describe how this name has miraculously contributed to the development of the subject, leading eventually to the opening, in 1991, of a new section in the *Mathematical Reviews* (03B53).

However, some criticisms have been addressed to this terminology; for instance, in his review of some of the papers included in [34], F. G. Asenjo notes: 'the expression "paraconsistent logic" ... is largely a misnomer. The term has a misleading public-relations timidity about it, having been introduced to avoid the arousing the negative reflexes triggered by the word "inconsistency" ' (see [6], p. 1503). This problem is certainly not a simple one, and is related to the philosophical problematic of knowing if paraconsistent logic is rival or complementary to classical logic. Some have clearly brought paraconsistent logic in the direction of a complementary logic, but the other trend is also a vivid one (cf. [34]). (See Section 5 below for a discussion.)

In his 1963 paper, da Costa presented the paraconsistent logic C_1 (and a related hierarchy of similar logics). Later, he would develop first-order calculi, description calculi and set theories based on these calculi. But he has formulated not only these paraconsistent logics of type C, but many others as well. He has also not worked alone, but has founded a 'school' of paraconsistent logic in Brazil. Moreover, he has worked with the Polish logicians J. Kotas and L. Dubikatjis, and with them has developed a systematic study of Jaskowski's problem (see, for instance [29]). With A. I. Arruda and L. Z. Puga, he has worked on a modern formalization of Vasil'év's system (see, for example, [36]).

Da Costa has not only contributed to the birth and development of paraconsistent logic as an autonomous field of research in mathematics by creating new systems, but also by organizing the subject, and tracing back its 'forerunners' and making their work better known.

The logic C_1 (and some connected ones) is one of the most studied and best known paraconsistent logic. A great number of papers have been published about it, especially (but not only) in *Notre Dame Journal of Formal Logic*. This logic was first presented as a system of deduction of a Hilbert-type (cf. [13]). A few years later, A. R. Raggio, in [38], tried to formulate a Gentzen-type sequent system, but did not succeed. It was only recently that J.-Y. Béziau gave such a version, with the corresponding cut-elimination theorem (see [9]). A semantics for C_1 would be provided only many years later, in [18]. The question of the algebraization of C_1 has also been the subject of many investigations. Da Costa has proposed a kind of algebraization as early as 1966 (see [14]), inspired by some ideas of H. Curry. Although some later results have shown that C_1 is not algebraizable, in the usual sense of the word (cf. [33] and [30]), da Costa's algebraic proposal can be defended in the face of Eytan's work (see [26]).

The idea of C_1 was to provide a logic which is paraconsistent, but as close as possible to classical propositional logic. If we consider the full usual language of propositional calculus (i.e. the connectives $\neg$, $\rightarrow$, $\wedge$, $\vee$), this means that all properties (independent of negation) of $\rightarrow$, $\wedge$, $\vee$ must be preserved. In other words, we want a conservative extension of the positive classical propositional calculus. On the other hand, the negation $\neg$ must be a *paraconsistent negation*, in the sense that there is at least a theory T from which we can deduce a formula a and its 'negation' $\neg a$ (T is *inconsistent*), but such that there is a formula b which cannot be deduced from T (T is *non-trivial*). The term *paraconsistent* is also used to qualify such a theory. Furthermore, this negation should, if possible, have all the properties of classical negation compatible with these features.

It is not easy to deal with this compromise. It is difficult indeed to analyze all the problems related to it. Notwithstanding this, we shall present here two possibilities. (1) A logic such as Johannson's minimal logic fits into this definition of paraconsistent logic, but no one wants to consider it as such; in fact, within this system, from an inconsistent theory it is possible to deduce the negation of any formula (to avoid this difficulty, Urbas has introduced the concept of strict paraconsistency; cf. [39]). (2) It seems that, on this basis, some strong meta-properties will never be met; such as, for instance, the replacement theorem.

The best to do is perhaps to study these problems in a concrete case. In C_1, the negation seems to be strong enough to deserve this name; for example, it obeys $\neg\neg a \rightarrow a$, $a \vee \neg a$, and one quarter of De Morgan's laws (see [8]). We may intend, however, to strengthen C_1 without reaching a logic such as Johannson's, where a contradiction relatively trivializes the system. The main drawback of C_1 is that the replacement theorem does not hold. Thus, we may intend to extend C_1 not only to get more theorems, but also to get more metatheorems (such as this one). However, this may turn out not to be possible; at least, no attempt on this direction has been successful until now.

Nevertheless, the logic C_1^+ presented here supplies a partial solution to this problem, for in it we can define a non trivial congruence relation, which is not the case in C_1 (as Mortensen has shown). On the other hand, this metatheorem is not reached by extending artificially C_1, but by strengthening it in a very natural way, and providing thus some interesting further theorems, specially another quarter of De Morgan's laws.

3 The paraconsistent logic C_1^+

In this section we present the paraconsistent propositional logic C_1^+ which is a strengthening due to J.-Y. Béziau (presented for the first time in [8]) of the paraconsistent logic C_1 of N. C. da Costa. (about C_1 see [13, 17, 18, 9]).

The basic idea of C_1 is that the weak paraconsistent negation $\sim$ has the following two properties: (1) $a \wedge \sim a$ obeys the principle of contradiction (this provides a strong classical negation $\neg$); and (2) if a and b obey the principle of contradiction, so do $\sim a, a \wedge b, a \vee b, a \supset b$. With (2), $\sim$ is not too weak and we have, for example, $\sim (a \wedge \sim b) \supset (\sim a \vee b)$.

We extend C_1 to C_1^+ by replacing (2) by the following condition (2'): if a *or* b obey the principle of contradiction, so do $\sim a, a \wedge b, a \vee b, a \supset b$. Then $\sim$ is stronger and we have, for example: $\sim (a \vee \sim b) \supset (\sim a \wedge b)$, which is false in C_1.

Remark 3.1

We have chosen here to use the symbol tilde to denote the paraconsistent negation and the usual symbol (in present day) $\neg$ to denote the classical negation definable in C_1^+ in terms of the weak one. This precaution has been taken in order not to confuse the reader by attributing a new signification to the usual symbol of negation. In general, in the literature, it is common to use the same symbol to denote classical negation, intuitionistic negation, minimal negation, etc., even if there is no absolute definition of what a negation is, and in particular it is not made explicit what the properties are which distinguish a negation from a unary operator such as a modal operator.

At the end of the preceding section, we have followed this way of approaching the issue, by using the usual name and symbol for negation to speak about a paraconsistent negation, even if it was not clear in which sense this negation deserves the name. An interesting attempt to deal with this problem is to be found in [39].

In a logic, there may be two operators or more, which deserves the name negation. Thus, of course, the qualification paraconsistent is to be relativized to each of these negations. In other words, a logic is paraconsistent if there is a negation in which it is a paraconsistent one.

REMARK 3.2

There are several ways to interpret the informal sentence 'a obeys the principle of contradiction'. One possibility is: '$\neg(a \wedge \neg a)$ is a thesis'; another one: 'from a and $\neg a$ we can derive anything'; a third one: 'if a is true, then $\neg a$ is false'. In C_1^+ these three interpretations are equivalent in the sense that using one or another, we get the same logic; the same arises in C_1. In [8] C_1 and C_1^+ are presented comparatively (with also some dual and other logics); there, (2) is called the *multiplicative law* and (2') the *additive law.* Correspondingly, C_1 is called Cx and C_1^+, $C+$. The aim of [8] was not to study C_1^+ for itself, but to present it among a bundle of possible logics that can be constructed extending the methods used by da Costa to develop C_1.

The negation in C_1^+ as in C_1 is neither truth-functional nor extensional (in the sense that the replacement theorems does not hold). However, a bivalent semantics can be provided. This kind of semantics was the birth of the *theory of valuation*, initiated by N. C. A. da Costa (see [19, 20]).

REMARK 3.3

It may seem strange to say that a logic in which the replacement theorem does not hold is not extensional because a lot of logics in which this theorem holds are called intensional (this is typically the case of the standard modal logics; see e.g. [27]). However, this terminology from a philosophical point of view seems inadequate and probably is the consequence of the confusion between truth-functionality and extensionality. This terminology is directly inspired by the one of Wojcicki. Furthermore, even a logic which is not extensional in our sense does not necessarily deserve the adjective 'intensional'; for example, we do not think that C_1^+ (or C_1) is an intensional logic (for a discussion on this topic see [11]).

3.1 Morphology

We consider an absolute free algebra of propositions: $\mathcal{P} = \langle P; \langle\sim; \wedge; \vee; \supset\rangle\rangle$ of type $\langle 1; 2; 2; 2\rangle$; $\sim$ will be called the *weak negation* and $\wedge, \vee$ and $\supset$ will be termed conjunction, disjunction and implication, respectively. $ATOM \subseteq P$ is the set of generators and the elements of $ATOM$ are called atomic propositions. We will write $a, b, \ldots$ elements of P; $T, T', \ldots$ subsets of P; $\Gamma, \Delta, \ldots$ finite subsets of P.

We define the following unary compound function (*strong negation*): $\neg a = \sim a \wedge \sim (a \wedge \sim a)$; and the function κ from P to P: $\kappa(a) = a$, for $a \in ATOM$, $\kappa(a \star b) = \kappa(a) \star \kappa(b)$, for $\star \in \{\wedge, \vee, \supset\}$, and finally $\kappa(\sim a) = \neg(\kappa(a))$. This last function is then extended to a function from the power set of P into itself: $\kappa(T) = \{\kappa(a), a \in T\}$.

We will use $k[a_1, \ldots, a_p]$ as an abbreviation for $a \wedge b$ (when it is not of the form $c \wedge \neg c$), or $a \vee b$ or $a \supset b$ or $\sim a$, that is to say, k is a connective and $a_1, \ldots, a_p$ are direct subpropositions of the proposition $k[a_1, \ldots, a_p]$.

3.2 Sequent calculus for C_1^+

3.2.1 The system S^+

Given Γ, Δ (finite subsets of P), we will use the following notation for the pair $\langle \Gamma; \Delta \rangle$: $\Gamma \rightarrow \Delta$. We will use the comma and the blank in the following usual way: we shall write, for example, $\rightarrow a, \Delta, \Delta'$ instead of $\emptyset \rightarrow \{a\} \cup \Delta \cup \Delta'$. We call PPC the positive propositional sequent calculus. The rules of S^+ are the rules of PPC (including the cut rule) plus the following rules:

$$\frac{\Gamma, a \rightarrow \Delta}{\Gamma \rightarrow \sim a, \Delta} \sim r \qquad \frac{\Gamma \rightarrow a \wedge \sim a, \Delta}{\Gamma, \sim (a \wedge \sim a) \rightarrow \Delta} \sim l$$

$$\frac{\Gamma \rightarrow k[a_1, \ldots, a_p], \Delta \qquad \Gamma', a_i, \sim a_i \rightarrow \Delta'}{\Gamma, \Gamma', \sim k[a_1, \ldots, a_p] \rightarrow \Delta, \Delta'} \sim kil$$

for each $i, 1 \leq i \leq p$.

HINT 3.4

The intuitive explanation of the rules $\sim kil$ is the following: the left right premise says that one of the direct subproposition a_i of $k[a_1, \ldots, a_p]$ 'obeys the principle of contradiction'. The rules say that in this condition the usual left rule for the negation holds for $k[a_1, \ldots, a_p]$; this can also be interpreted as $k[a_1, \ldots, a_p]$ 'obeys the principle of contradiction'. In fact these rules are equivalent to the axioms:

$$\sim (a_i \wedge \sim a_i) \supset \sim (k[a_1, \ldots, a_p] \wedge \sim k[a_1, \ldots, a_p])$$

This equivalence can be proved in a very similar way as the one presented in [7] for other analogue systems.

For the binary connectives there are thus two $\sim l$ rules; for $\sim$, there is only one:

$$\frac{\Gamma \rightarrow \sim a, \Delta' \qquad \Gamma', a, \sim a \rightarrow \Delta'}{\Gamma, \Gamma', \sim\sim a \rightarrow \Delta, \Delta'} \sim\sim l$$

PROPOSITION 3.5

The rule $\sim\sim l$ is equivalent in PPC to the following rule:

$$\frac{\Gamma, a \rightarrow \Delta}{\Gamma, \sim\sim a \rightarrow \Delta}$$

PROOF. Immediate using $\sim r$ and the cut rule. ∎

THEOREM 3.6
The following schemes are valid in C_1^+:

$$\begin{array}{rcl@{\qquad}rcl}
\sim(a\wedge b) & \supset & (\sim a\vee\sim b) & \sim(a\vee b) & \supset & (\sim a\wedge\sim b)\\
\sim(\sim a\wedge\sim b & \supset & (a\vee b) & \sim(\sim a\vee\sim b) & \supset & (a\wedge b)\\
\sim(a\wedge\sim b) & \supset & (\sim a\vee b) & \sim(a\vee\sim b) & \supset & (\sim a\wedge b)\\
\sim(\sim a\wedge b) & \supset & (a\vee\sim b) & \sim(\sim a\vee b) & \supset & (a\wedge\sim b)
\end{array}$$

REMARK 3.7
The schemes on the right are not valid in C_1.

PROOF. We give just one example; the others follow in a similar way:
We have:

$$\dfrac{\dfrac{\dfrac{a\ \to\ a,\sim b}{\to\ a,\sim a,\sim b}\,\sim r}{\to\ a\vee\sim b,\sim a}\,\vee r \qquad a,\sim a\ \to\ \sim a}{\sim(a\vee\sim b)\ \to\ \sim a}\,\sim\vee 1l \tag{3.1}$$

and

$$\dfrac{\dfrac{\dfrac{b\ \to\ a,b}{\to\ a,\sim b,b}\,\sim r}{\to\ a\vee\sim b,b}\,\vee r \qquad \dfrac{\dfrac{\dfrac{b\ \to\ b}{\to\ \sim b,b}\,\sim r \qquad b,\sim b\to b}{\sim\sim b\ \to\ b}\,\sim\sim l}{\sim\sim b,\sim b\ \to\ b}\,\sim l}{\sim(a\vee\sim b)\ \to\ b}\,\sim\vee 2l \tag{3.2}$$

Then, from the proofs (1) and (2) we can proceed as follows:

$$\dfrac{\dfrac{\sim(a\vee\sim b)\ \to\ \sim a \qquad \sim(a\vee\sim b)\ \to\ b}{\sim(a\vee\sim b)\ \to\ \sim a\wedge b}\,\wedge r}{\to\sim(a\vee\sim b)\ \supset\ (\sim a\wedge b)}\,\supset r$$

obtaining in this way the desired sequent. ∎

3.2.2 Strong negation and translation to classical logic

THEOREM 3.8
In *PPC*, $\sim r$ is equivalent to:

$$\dfrac{\Gamma,a\ \to\ \Delta}{\Gamma\ \to\ \neg a,\Delta}\,\neg r$$

and $\sim l$ is equivalent to:

$$\dfrac{\Gamma\ \to\ a,\Delta}{\Gamma,\neg a\ \to\ \Delta}\,\neg l$$

PROOF. Immediate. ∎

COROLLARY 3.9
$T \vdash_K a$ if and only if $\kappa(T) \vdash_{C^+} \kappa(a)$, ($\vdash_K$ being classical logic and $\vdash_{C^+}$ the logic C_1^+ generated by the system S_1^+).

COROLLARY 3.10
Let S be an ordinary metastatement about $\vdash_K$, then:

$$S \text{ if and only if } \kappa(S)\ (\vdash_{C^+})$$

where $\kappa(S)$ os a statement constructed from S replacing a by $\kappa(a)$ and T by $\kappa(T)$ for every a and T in S.

3.2.3 Cut elimination and corollaries

THEOREM 3.11
Cut elimination holds for C_1^+.

PROOF. Similar as the proof for C_1. See [9]. ■

COROLLARY 3.12
C_1^+ is decidable.

PROOF. We define the *sphere* of a proposition a: sph $(a) = \{\ b$: b is a subproposition of a or the negation of a proper subproposition of $a\}$; and the sphere of a set of propositions: sph $(T) = \{$sph (a): $a \in T\ \}$. In a cut-free proof of a sequent $\langle \Gamma; \Delta \rangle$, it is easy to see that the only propositions that appear are proposition of the sphere of $\Gamma \cup \Delta$. ■

3.3 Semantics

3.3.1 Bivalent non truth-functional semantics

We consider the following set of bivaluations $\mathcal{V}^+ \subseteq \{0,1\}^{\mathcal{P}}$:

$\delta \in \mathcal{V}^+$ if and only if:

$[\wedge] : \delta(a \wedge b) = 1 \Leftrightarrow \delta(a) = 1$ and $\delta(b) = 1$
$[\vee] : \delta(a \vee b) = 1 \Leftrightarrow \delta(a) = 1$ or $\delta(b) = 1$
$[\supset] : \delta(a \supset b) = 1 \Leftrightarrow \delta(a) = 0$ or $\delta(b) = 1$
$[\sim r] : \delta(a) = 0 \Rightarrow \delta(\sim a) = 1$
$[\sim l] : \delta[(a \wedge \sim a)] = 1 \Rightarrow \delta[\sim (a \wedge \sim a)] = 0$
$[\sim kil]$: If $\delta(k[a_1, \ldots, a_p]) = 1$ and $\delta(a_i) = 0$ or $\delta(\sim a_i) = 0$ for one $i, (1 \leq i \leq p)$, then $\delta(\sim k[a_1, \ldots, a_p]) = 0$

THEOREM 3.13
$[\sim r]$ is equivalent to $[\neg r]$ and $[\sim l]$ is equivalent to $[\neg l]$ (where $[\neg r]$ and $[\neg l]$ are the usual conditions for classical negation).

PROOF. The same as for C_1. See [9]. ■

The semantical deductibility relation $\vdash_{\mathcal{V}^+}$ is defined as usual.

3.3.2 Trivalent non truth-functional semantics

We consider a set of three values $\{\langle 0;1\rangle, \langle 1;0\rangle, \langle 1;1\rangle\}$ (where $\langle 1;0\rangle$ and $\langle 1;1\rangle$ are both designated values) and we define the set $\mathcal{V}_3^+$ of functions from P to this set. We shall give just, as an example, the conditions for the conjunction:

a	b	$a \wedge b$
$\langle 0;1\rangle$	$\langle 0;1\rangle$	$\langle 0;1\rangle$
$\langle 0;1\rangle$	$\langle 1;0\rangle$	$\langle 0;1\rangle$
$\langle 0;1\rangle$	$\langle 1;1\rangle$	$\langle 0;1\rangle$
$\langle 1;0\rangle$	$\langle 0;1\rangle$	$\langle 0;1\rangle$
$\langle 1;0\rangle$	$\langle 1;0\rangle$	$\langle 1;0\rangle$
$\langle 1;0\rangle$	$\langle 1;1\rangle$	$\langle 1;0\rangle$
$\langle 1;1\rangle$	$\langle 0;1\rangle$	$\langle 0;1\rangle$
$\langle 1;1\rangle$	$\langle 1;0\rangle$	$\langle 1;0\rangle$
$\langle 1;1\rangle$	$\langle 1;1\rangle$	$\langle 1;0\rangle$
$\langle 1;1\rangle$	$\langle 1;1\rangle$	$\langle 1;1\rangle$

The table presented here is a generalized truth table in the sense of [18]. We will present such kind of tables in a more precise manner for the bivalent semantics in Section 3.5.1 below. Each line must be interpreted as a condition the elements of $\mathcal{V}_3^+$ should satisfy; i.e. for example, the first line of the table states that an element t of $\mathcal{V}_3^+$ must satisfy the following condition: for every a and b, $t(a) = \langle 0:1\rangle$ and $t(b) = \langle 0;1\rangle$ implies that $t(a \wedge b) = \langle 0;1\rangle$.

For a discussion about this non truth-functional three-valued semantics and the usual many-valued semantics, the reader may consult [21].

The advantage of this semantics is that the values of a compound proposition depends, as in the classical case, only on its subpropositions.

Thus we have here an example of the usefulness of increasing the number of values even in a non truth-functional context.

It is out of the scope of this paper to present the abstract result of [10]. The idea, roughly speaking, is to preserve rightly the transformation set of designated values. This kind of transformation has been used in the case of matrix theory of Suzko to show how to reduce any many-valued matrix semantics into a bivalent (non truth-functional) semantics (see [31] pp. 72–73). J.-Y. Béziau in the above referred paper has extended this method to reduce a wider range of semantics (not necessarily given by matrix) to bivalent semantics. To do so he has established a more general result saying that if two semantics are epimorphic they induce the same logic (given a very general definition of semantics and a corresponding notion of epimorphism). Furthermore, it is possible to use this result in a 'reverse' way, as in the present case, to increase the number of values.

THEOREM 3.14
$T \vdash_{\mathcal{V}^+} a \;\Leftrightarrow\; T \vdash_{\mathcal{V}_3^+} a$.

PROOF. We consider the function ϕ from $\mathcal{V}^+$ to $\mathcal{V}_3^+$:

$$\begin{array}{lcl} \phi(\delta)(a) = \langle 0;1\rangle & \Leftrightarrow & \delta(a) = 0 \text{ and } \delta(\sim a) = 1 \\ \phi(\delta)(a) = \langle 1;0\rangle & \Leftrightarrow & \delta(a) = 1 \text{ and } \delta(\sim a) = 0 \\ \phi(\delta)(a) = \langle 1;1\rangle & \Leftrightarrow & \delta(a) = 1 \text{ and } \delta(\sim a) = 1 \end{array}$$

Then we use the abstract result about semantical epimorphisms of [10]. ∎

3.4 Completeness

The completeness theorem can be proved exactly in the same way as in the case of C_1 (see [18]), the adaptation of the proof being very small if we first transform the system S^+ into a Hilbert-type system, and this can be done also in a very similar way as in the case of the sequent system for C_1 presented in [9].

However, J.-Y. Béziau has recently proved a general result on the connection between sequent rules and conditions for bivaluations which, applying in a particular case like the ones of $\mathcal{V}^+$ and S^+, gives instantaneously the completeness theorem (see e.g. [12]).

It is out of the scope of this paper to present such a result, but we will say a word about it: this result shows that in the case of some particular sequent systems (those which are structurally standard in the sense that they contain the usual standard rules and that there is no particular restriction on the form of the sequent as in the case of the intuitionistic system LJ), the natural interpretation in terms of truth and falsity always holds. As the reader may check, the conditions for the bivaluations $[\sim kil]$ are the direct translation, under this natural interpretation, of the rules $\sim kil$.

3.5 Truth tables

3.5.1 Construction of truth tables

Given a proposition a, the truth table of a is constructed in the following way:

- Write on the first line, all the propositions of the sphere of a, ordering them according to their complexity;
- Let $n \in \mathbb{N}^*$ be the number of atomic propositions of a, write the n first columns as in the classical case;
- the $n+m$ ($m \in \mathbb{N}$) first columns being completed, write the next column according to the following instructions:
  1. If the proposition on the top is not a negation, proceed as in the classical case;
  2. If not, the proposition is of the form $\sim b$; then:
     - Fill the existing lines as in the classical case;
     - Among the lines where 1 is written on the column whose top is b, rewrite the following lines writing 1 on the column whose top is $\sim b$;
     - If $b \in ATOM$, rewrite all these lines;
     - If $b = k[a_1, \ldots, a_p]$, rewrite the lines where the value is 1 for both columns whose tops are a_i and $\sim a_i$, for every i, $1 \leq i \leq p$.

THEOREM 3.15 (SOUNDNESS AND COMPLETENESS OF THE METHOD)
Given a truth table for a proposition b:

- Each of the functions defined by a line of the truth table can be extended to a function from P to $\{0, 1\}$ which is a member of $\mathcal{V}^+$.
- Given a function $\delta \in \mathcal{V}^+$, its restriction to the sphere of b is identical to the function defined by a line of the truth table.

PROOF. The proof is carried out in a similar way to the proof for C_1; see for example [18]. ∎

COROLLARY 3.16
C_1^+ is semantically decidable.

3.5.2 Applications of the method

THEOREM 3.17
The following schemes are not valid in C_1^+:

$$\begin{array}{ccc} (\sim a \wedge \sim b) & \supset & \sim (a \vee b) \\ (a \wedge b) & \supset & \sim (\sim a \vee \sim b) \\ (\sim a \wedge b) & \supset & \sim (a \vee \sim b) \\ (a \wedge \sim b) & \supset & \sim (\sim a \vee b) \end{array} \qquad \begin{array}{ccc} (\sim a \vee \sim b) & \supset & \sim (a \wedge b) \\ (a \vee b) & \supset & \sim (\sim a \wedge \sim b) \\ (\sim a \vee b) & \supset & \sim (a \wedge \sim b) \\ (a \vee \sim b) & \supset & \sim (\sim a \wedge b) \end{array}$$

PROOF. We give just one example:

a	b	$\sim a$	$\sim b$	$a \vee \sim b$	$\sim (a \vee \sim b)$	$\sim a \wedge b$	$(\sim a \wedge b) \supset \sim (a \vee \sim b)$	
0	0	1	1	1	0	0	1	
0	1	1	0	0	1	1	1	K
1	0	0	1	1	0	0	1	
1	1	0	0	1	0	0	1	
0	1	1	1	1	0	1	0	
1	0	1	1	1	0	0	1	
1	1	0	1	1	0	0	1	$\mathcal{V}^+$
1	1	1	0	1	0	1	0	
1	1	1	1	1	0	1	0	
1	1	1	1	1	1	1	1	

∎

THEOREM 3.18
None of the following forms of contraposition is valid in C_1^+: $(a \supset b) \supset (\sim b \supset \sim a)$, $(\sim a \supset b) \supset (\sim b \supset a)$, $(a \supset \sim b) \supset (b \supset \sim a)$ and $(\sim a \supset \sim b) \supset (b \supset a)$.

PROOF. We give just one example, constructing the truth table of $(a \supset b) \supset (\sim b \supset \sim a)$:

a	b	$\sim a$	$\sim b$	$a \supset b$	$\sim b \supset \sim a$	$(a \supset b) \supset (\sim b \supset \sim a)$	
0	0	1	1	1	1	1	
0	1	1	0	1	1	1	K
1	0	0	1	0	0	1	
1	1	0	0	1	1	1	
0	1	1	1	1	1	1	
1	0	1	1	0	1	1	
1	1	0	1	1	0	0	$\mathcal{V}^+$
1	1	1	0	1	1	1	
1	1	1	1	1	1	1	

∎

COROLLARY 3.19
The replacement theorem does not hold for C_1^+.

C. Mortensen has shown that in C_1 it is not possible to define an equivalence relation which is not the identity (see [33]). C_1^+ has the very interesting feature of having a non-trivial congruence that we call well-equivalence.

We say that a and b are logically equivalent (notation: $a \equiv b$) when for every $\delta \in \mathcal{V}^+$, $\delta(a) = \delta(b)$.

A proposition a is said to be *well-behaved* when for every $\delta \in \mathcal{V}^+$, $\delta(a) = 1 \Rightarrow \delta(\sim a) = 0$

Now we say that a and b are *well-equivalent* (notation: $a \simeq b$) if and only if they are both well-behaved and if they are logically equivalent.

LEMMA 3.20
If a proposition has a well-behaved subproposition, then it is well-behaved.

PROOF. The proof is carried out by induction on the complexity of a proposition a. The case where a is an atom trivially holds because it has no well-behaved subproposition. If a is of the form $b \wedge \neg b$, it also trivially holds because a is well-behaved.

Now suppose that $a = k[a_1, \ldots, a_p]$. If a has a well-behaved subproposition, it is either a direct subproposition of a or it is a subproposition of, at least, one $a_i (1 \leq i \leq p)$, then by the induction hypothesis there is, at least, one direct subproposition of a which is well-behaved. Thus, if a has a well-behaved subproposition, it has a direct well-behaved subproposition, from which we can conclude (by [$\sim$kil]) that a is well-behaved. ∎

THEOREM 3.21
The relation of well-equivalence is a congruence relation for C_1^+.

PROOF. Given a proposition c and two propositions a and b such that a is a proper subproposition of c and such that $a \simeq b$, we will prove by induction on the complexity of c that: $c[a] \equiv c[b/a]$, where $c[b/a]$ is the proposition that we get replacing a by b in c.

i) If c is an atom, the property vacuously holds.

ii) c is a complex proposition of the form $\sim d$.
By the lemma just proved, $d[a]$ is well-behaved; thus, we can apply the induction hypothesis and we will have: $d[a] \equiv d[b/a]$. By the same lemma, we also know that $d[b/a]$ is well-behaved.
Now suppose that there exists $\delta \in \mathcal{V}^+$ such that $\delta(\sim d[a]) = 0$ and $\delta(\sim d[b/a]) = 1$ (the other case is left to the reader); thus, by [$\sim$r], $\delta(d[a]) = 1$ and $\delta(d[b/a]) = 0$ because $d[b/a]$ is well-behaved, but this is absurd because $d[a] \equiv d[b/a]$.

iii) The other three cases are easy and are left to the reader.

∎

4 Applications of paraconsistent logic

4.1 The usefulness of paraconsistent reasoning

There are many useful applications of paraconsistent logic for all kinds of reasoning, artificial or natural. The advantages of employing this logic are obvious: in the presence of a contradiction, we can go on making interesting reasonings (especially using a very strong paraconsistent logic like C_1^+) without, as in classical logic, (1) supposing that one of the terms of the contradiction must be rejected or (2) deriving anything. Obviously, (2) is the dead end of logical reasoning, but (1) is, without luck, not better, for if we make the wrong choice there is no way out. And in many cases it is simply not possible to know which is the wrong choice and which is the right one. Maybe there is, *in reality*, no contradiction. But it seems that contradictions are enclosed in any important amount of complex knowledge. Perhaps these contradictions are only *fictitious contradictions*. But even if there is, now, in the present state of affairs, no way to solve these contradictions, it may be useful to deal with them, without the peril of direct triviality.

We shall give here concrete examples of reasoning involving contradictions. We take the example of medicine, which has already been investigated in [23] using another paraconsistent logic different from the ones presented in this paper. Medicine is a subject of special interest, because it is not rare that two physicians give two different contradictory diagnostics for the same observable symptoms. This is not necessarily due to the incompetence of the physicians, but rather to the complexity of medicine. And thus this situation can also arise in the case of a *MES (Medical Expert System)*.

4.2 Examples of paraconsistent reasonings in Medicine

John Smith is sick; he goes to Dr. Bouvard and this one tells him that he has got cancer. John Smith decides to consult another specialist, namely Dr. Pécuchet, who turns out to be categorical: he tells him that he has not got cancer.

Dr. Pécuchet does not agree with his colleague on this point and many others but there is at least one thing that they both recognize:

If John Smith has got cancer he will die in the next three months.

John Smith is very bewildered, he thinks one of them must be wrong, but he does not know which one, because Dr. Bouvard is a very renowned physician and so is Dr. Pécuchet.

Now we will show that if John Smith uses paraconsistent logic, he can make interesting reasonings, without supposing that Dr. Pécuchet or Dr. Bouvard is wrong.

Reasoning 1:

We will show that from the statement of Dr. Bouvard, the statement of Dr. Pécuchet and the statement they both agree on we cannot infer that:

If John Smith has not got cancer he will not die in the next three months.

Let us use the following transcriptions:

(a) John Smith has got cancer.

(b) John Smith will die in the next three months.

Thus we will show that: $\{a, \sim a, a \supset b\} \nvdash_{C_1^+} \sim a \supset \sim b$.

PROOF. There exists $\delta \in \mathcal{V}^+$ such that $\delta(a) = \delta(\sim a) = \delta(a \supset b) = 1$ and $\delta(\sim a \supset \sim b) = 0$:

a	$\sim a$	b	$\sim b$	$a \supset b$	$\sim a \supset \sim b$
1	1	1	0	1	0

∎

REMARK 4.1
This proof is valid in C_1 and even in the fragment C_i (see [9]) of C_1.

Now, we are going to give an example of reasoning typical of C_1^+.

Reasoning 2:
Suppose Dr. Bouvard says:
It is not possible that:
John Smith has not got cancer
and
John Smith will die in the next three months.

From this statement (and only this one) we will show that we can infer, as in classical logic, that:
If John Smith has not got cancer he will not die in the next three months.

PROOF. We can interpret '*it is not possible that d*' as '*there exists δ such that $\delta(d) = 0$*' (see Wittgenstein's *Tratactus*, 4.464 /5.525).

For every $\delta \in \mathcal{V}^+$, we have $\delta(\sim a \wedge b) = 0$; using [$\sim$r], we get: for every $\delta \in \mathcal{V}^+$, $\delta(\sim (\sim a \wedge b)) = 1$. Using the completeness theorem, we have: $\vdash_{C_1^+} \sim (\sim a \wedge b)$. We shall now give a proof of the sequent $\sim (\sim a \wedge b) \to \sim a \supset \sim b$ in C_1^+.

$$\dfrac{\dfrac{\sim a \to \sim a \qquad \dfrac{b \to b}{\to b, \sim b}\,\sim r}{\sim a \to (\sim a \wedge b), \sim b}\,\wedge r \qquad b, \sim b \to \sim b}{\dfrac{\sim (\sim a \wedge b), \sim a \to \sim b}{\sim (\sim a \wedge b) \to \sim a \supset \sim b}\,\supset r}\,\sim\wedge 2l$$

Thus we have $\{\sim (\sim a \wedge b)\} \vdash_{C_1^+} \sim a \supset \sim b$, from which we can conclude that $\vdash_{C_1^+} \sim a \supset \sim b$. ∎

REMARK 4.2
This reasoning is not only typical of C_1^+ by contrast to C_1, in the sense that it is not valid in C_1; but also by contrast to classical logic, in the sense that if we are in the presence of contradictions, there is no trivialization. That is to say, if we have furthermore a contradictory hypothesis, from the point of view of the weak negation $\sim$, such as $\vdash_{C_1^+} c \wedge \sim c$; for example the sentence *Dr. Bouvard is a quack and Dr. Bouvard is not a quack.*

What we can say about these two examples of reasoning is that, in presence of contradictions, C_1^+ does not allow us to do some reasoning which is not valid without contradictions (cf. Reasoning 1: we can not deduce $\sim a \supset \sim b$ from $a \supset b$ in classical logic and it is not possible to do so in presence of a set of contradictory hypotheses such as $\{a, \sim a\}$), but C_1^+ allows us to do a good part of classical reasoning in presence of contradictions without reaching triviality (Reasoning 2).

5 Some philosophical remarks

From a philosophical perspective, just as from a technical one, paraconsistent logic has also given rise to various, interesting considerations. However, by no means shall a comprehensive approach be proposed here. The following brief remarks are devised only to acquaint the reader with some already explored lines, leaving to further works a more detailed approach (for additional philosophical considerations, see [15], [35] and [37]).

At the outset, it should be clear that, despite being a non-classical logic, paraconsistent logic, from our viewpoint, does not constitute a tentative approach to challenge classical, standard logical conceptions – whose domain and main features are assured beyond any doubts. Rather, it was mainly devised in order to supply alternative tools, not found in the extant formalisms, so that some specific mathematical and logical problems, not possibly addressed to within a classical framework, could be reasonably considered. For instance, in order to study semantical paradoxes in set theory, taking them at face value, and not trying (as usual) only to evade them, some paraconsistent machinery is called for. Similarly, the analysis of specific principles in their full cogency – as, for example, the principle of comprehension in set theory or in higher order predicate logic – also demands considerations on paraconsistent grounds. Or else, a deeper understanding of some concepts, such as of negation, can be better attained through the employment of a paraconsistent framework ([3], pp. 11–12).These are just some possible examples to illustrate our point; there are, though, still many others that could be mentioned, if it were necessary.

More importantly, however, is perhaps to stress, once again, the parallel, from a conceptual point of view, between the creation of Non-Euclidean geometry and the rise of paraconsistent logic. Undeniably, both of them present, within mathematical and logical contexts, unquestionable relevance. Nevertheless, in addition to this, and opposed on this regard to many other scientific theories, their philosophical significance is also incontestable. In effect, if non-classical geometries have afforded us a deeper understanding of some scientific notions, non-classical logics in general, and paraconsistent logic in particular, have provided a crucial theoretical setting to grasp the true meaning of logicity. In both cases, the gain is enormous.

It also deserves to be remarked that paraconsistent logic can be further regarded (within the context of its applications as well as of its interpretation) from two different points of view: (1) as a logic complementary to classical logic (for instance, by interpreting its negations as connectives more general than classical negation); or (2) as a kind of heterodox logic, incompatible with classical logic, whose destiny is to replace the latter in all or some of its *applications*.

Determining which of these possibilities is the case *in general*, however, is something rather delicate. Indeed, such a question seems not to present a direct answer, for paraconsistent logic is not a uniform subject; we can, in fact, construct rival as well complementary paraconsistent logics. Moreover, to a certain extent, this issue cannot be established only through the resources supplied by logic and mathematics alone, but further philosophical considerations (related, in particular, to the philosophy of sciences) are needed as well. Notwithstanding this, *particular* cases can be frequently analyzed.

For instance, with regard to C_1^+, we should remark, it can be viewed as complementary to classical propositional logic, exactly in the same sense as a modal logic, such as e.g. $S4$, is complementary to this very logic. Suppose that we have the usual absolutely free algebra of propositions $\mathcal{P} = \langle P; \langle \rightarrow; \wedge; \vee; \neg \rangle \rangle$, and the usual corresponding classical notion of deduction $\vdash_K$. Now, consider the algebra $\mathcal{P}(\sim) = \langle P; \langle \rightarrow; \wedge; \vee; \sim \rangle \rangle$, and suppose that $\sim$ obeys the rules of the weak negation of C_1^+. C_1^+ is a conservative extension of K, in the sense that the restriction of $\vdash_{C_1^+}$ to $\mathcal{P}$ is $\vdash_K$. But the interesting point to know is whether, when defining the strong negation as we have done, we get the 'same' negation as the one which was already there. In C_1, $\neg a$ is logically equivalent to $\sim a \wedge \sim (a \wedge \sim a)$. But this property is not very satisfactory, for we cannot replace, in C_1, one of this proposition by the other; for instance, $\sim \neg a$ is not logically equivalent to $\sim (\sim a \wedge \sim (a \wedge \sim a))$. But in C_1^+, we can accomplish this, using the fact that $\neg a$ and $\sim a \wedge \sim (a \wedge \sim a)$ are congruent formulas, as defined in 3.5.2.

Therefore, the conclusion is that C_1^+ is really an extension of classical logic, which (in this sense) complements it. Within C_1^+, we can deal with all we have been dealing with in classical logic, with no alterations, and furthermore, we can use a weak negation which permits us to make some extra patterns of reasoning as well.

However, we should note that, in general, the very distinction between rivalry and complementarity in connection to non-classical logics is something not precise, and a detailed examination of this issue depends on the particular formulations such a distinction receives. There are certainly circumstances, nevertheless, in which one can employ paraconsistent logic, and in which classical logic cannot be applied (for instance, in the semantical analysis of some paradoxes, such as Russell's: Russell's set *exists* in some paraconsistent set theories). In this sense, paraconsistent logic is rival to classical one.

The big question, however, is to know whether our world is in fact contradictory or not, and such a question was not definitively answered yet.

Anyway, in connection to this question, there are those who claim that paraconsistent logic is *rival* mainly because of the existence of some 'true contradictions' (cf. [34]). But what are the exact connections between the problematic rival/complementary and the belief in the existence or non-existence of contradictions? For instance, Priest and Routley seem to believe that true contradictions entail that paraconsistent logic is a rival logic. As far as we see it, nonetheless, and in conformity with what we have just remarked, independently of a commitment to this kind of contradiction, paraconsistent logic may be conceived, in certain contexts, as a rival one. However, we shall not pursue this issue further here.

A final word. We should note that there are infinitely many non-equivalent paraconsistent logical systems. In this respect, the situation is similar to that of modal logic. On the other hand, as it was already mentioned above, paraconsistent logic has given rise to various important developments in philosophy, mathematics, the empirical sciences as well as technology.

For details, one may consult, for instance, [3, 25] (both of these interesting and highly informative papers – which were extensively used in order to articulate the historical outline presented in Section 1 above as well as the present philosophical remarks – contain a rather detailed list of bibliographical references) [34, 2, 28, 32]. For an overview, see [22].

Acknowledgements

We would like to thank Marcelo Tsuji for preparing the final LaTeXversion of the paper, as well as three anonymous referees for their important remarks on an earlier version of it.

The second author is supported by a grant from the Swiss National Research Fund.

References

[1] Alves, E. 'The first axiomatization of a paraconsistent logic', *Bulletin of the Section of Logic* **21**:19–20, 1992.

[2] Arruda, A. "On the Imaginary Logic of N. A. Vasil'év', in Arruda, A. *et* al. (eds.), *Non-Classical Logics, Model Theory and Computability*, Amsterdam, North-Holland (1977), pp. 3–24.

[3] Arruda, A. 'A survey of Paraconsistent Logic', in Arruda, A. *et* al. (eds.), *Mathematical Logic in Latin America*, Amsterdam, North-Holland (1980), pp. 1–41.-

[4] Arruda, A., da Costa, N. C. A. and Chuaqui, R. (eds.), *Non-Classical Logics, Model Theory and Computability*, Amsterdam, North-Holland (1977).

[5] Arruda, A., Chuaqui, R. and da Costa, N. C. A. (eds.), *Mathematical Logic in Latin America*, Amsterdam, North-Holland (1980).

[6] Asenjo, F.G. 'Review of Priest et al. (eds.), *Paraconsistent Logic: Essays in the Inconsistent*', *The Journal of Symbolic Logic* **56**:1503, 1991.

[7] Béziau, J. Y. 'Calculs des séquents pour logique non-alètique', *Logique et Analyse* **125–126**:143–155, 1989.

[8] Béziau, J. Y. 'Logiques construites suivant les méthodes de da Costa I', *Logique et Analyse* **131–132**:259–272, 1990.

[9] Béziau, J. Y. 'Nouveaux résultats et nouveau regard sur la logique paraconsistent C_1', *Logique et Analyse* **137–138**, 1991.

[10] Béziau, J. Y. 'Recherches sur la logique abstraite: les logiques normales', *Acta Universitatis Wratislaviensis, Serie Logika* **16**, 1994.

[11] Béziau, J. Y. 'Du Pont's paradox and the problem of intensional logic', in Kólař, P. and Svoboda, V. (eds.), *Logica' 93 - Proceedings of the 7th International Symposium*, Prague, Academy of Sciences of the Czech Republic (1994).

[12] J. Y. Béziau. 'Théorie de la valuation', Appendix 2 in da Costa, N. C. A., *Logiques Classiques et Non Classiques*, Paris, Masson (1995).

[13] da Costa, N. C. A. 'Calculs propositionnels pour les systèmes formels inconsistants', *C. R. de l'Acadèmie des Sciences de Paris* **257**:3790–3793, 1963.

[14] da Costa, N. C. A. *Álgebras de Curry*, Sao Paulo, IME-USP (1966).

[15] da Costa, N. C. A. 'The Philosophical Import of Paraconsistent Logic', *The Journal of Non-Classical Logic* **1**:1–19, 1982.

[16] da Costa, N. C. A. 'La filosofía da la lógica de Francisco Miró Quesada', in Sobrevilla, D. and Belaunde, D. G. (eds.), *Lógica, Razón y Humanismo*, Lima, Universidad de Lima (1992).

[17] da Costa, N. C. A. *Sistemas Formais Inconsistentes*, Curitiba, Editora Universidade Federal do Paraná (1993).

[18] da Costa, N. C. A. and Alves, E. H. 'A semantic Analysis of the Calculi C_n', *Notre Dame Journal of Formal Logic* **16**:621–630, 1977.

[19] da Costa, N. C. A. and Béziau, J. Y. 'La théorie de la valuation en question' in Abad, M. (ed.), *Proceedings of the Ninth Latin American Symposium on Mathematical Logic*, Bahia Blanca, Universidad del Sur, 1994, pp. 95–104.

[20] da Costa, N. C. A. and Béziau, J. Y. *Théorie de la valuation*, to appear.

[21] da Costa, N. C. A., Béziau, J. Y. and Bueno, O. A. S. 'Review-essay of G. Malinowski, *Many-Valued Logics*', to appear in *Modern Logic*.

[22] da Costa, N. C. A. and Marconi, D. 'An Overview of Paraconsistent Logic in the 80's', *The Journal of Non-Classical Logic* **6**:5–31, 1989.

[23] da Costa, N. C. A. and Subrahmanian, V. S. 'Paraconsistent logics as a formalism for reasoning about inconsistent knowledge bases', *Artificial Intelligence in Medicine* **1**:167-174, 1989.

[24] Došen, K. 'The first axiomatization of relevant logic', *Journal of Philosophical Logic* **21**:339–356, 1992.

[25] D'Ottaviano, I. 'On the development of paraconsistent logic and da Costa's work', *Journal of Non-Classical Logic* **7**:9–72, 1990.

[26] Eytan, M. 'Tableaux de Smullyan, ensembles de Hintikka et tout ça: un point de vue algébrique', *Mathém., Sci. Hum.* **48**:21–27, 1974.

[27] Gabbay, D. M. 'What is negation in a system?' in Drake, F. R. and Truss, J. K. (eds.), *Logic Colloquium'86*, Amsterdam, North-Holland, 1988, pp. 95–112.

[28] Grana, N. *Logica Paraconsistente*, Naples, Loffredo (1983).

[29] Kotas, J. and da Costa, N. C. A. 'On some modal systems defined in connection with Jaskowski's problem', in Arruda, A. *et* al. (eds.), *Non-Classical Logics, Model Theory and Computability*, Amsterdam, North-Holland, 1977, pp. 57–73.

[30] Lewin, R. A., Mikenberg, I. S. and Schwarz, M. G. 'C_1 is not algebraizable', *Notre Dame Journal of Formal Logic* **32**:609–611, 1991.

[31] Malinowski, G. *Many-Valued Logics*, Oxford, Clarendon-Press (1993).

[32] Marconi, D. *La Formalizzazione della Dialettica*, Turin, Rosenberg *et* Sellier (1979).

[33] Mortensen, C. 'Every quotient algebra for C_1 is trivial', *Notre Dame Journal of Formal Logic* **21**:694–700, 1980.

[34] Priest, G., Routley, R. and Norman, J. (eds.) *Paraconsistent Logic: Essays on the Inconsistent*, Munich, Philosophia Verlag (1989).

[35] Priest, G. and Routley, R. 'The philosophical significance and inevitability of inconsistency', in Priest *et* al. (eds.), *Paraconsistent Logic: Essays on the Inconsistent*, Munich, Philosophia Verlag (1989).

[36] Puga, L. Z. and da Costa, N. C. A. 'On the imaginary logic of N. A. Vasil'év', *Zeitschrift für Mathematische Logik ung Grunglagen der Mathematik* **34**:205–211, 1988.

[37] Quesada, F. M. 'Paraconsistent logic: some philosophical issues', in Priest *et* al. (eds.), *Paraconsistent Logic: Essays on the Inconsistent*, Munich, Philosophia Verlag (1989).

[38] Raggio, A. R. 'A propositional sequence-calculi for inconsistent systems', *Notre Dame Journal of Formal Logic* **9**:359–366, 1968.

[39] Urbas, I. 'Paraconsistency', *Studies in Soviet Thought* **39**:343–354, 1990.

Received 30 August 1994. Revised 26 May 1995