Is the principle of contradiction a consequence of $x^2 = x$?

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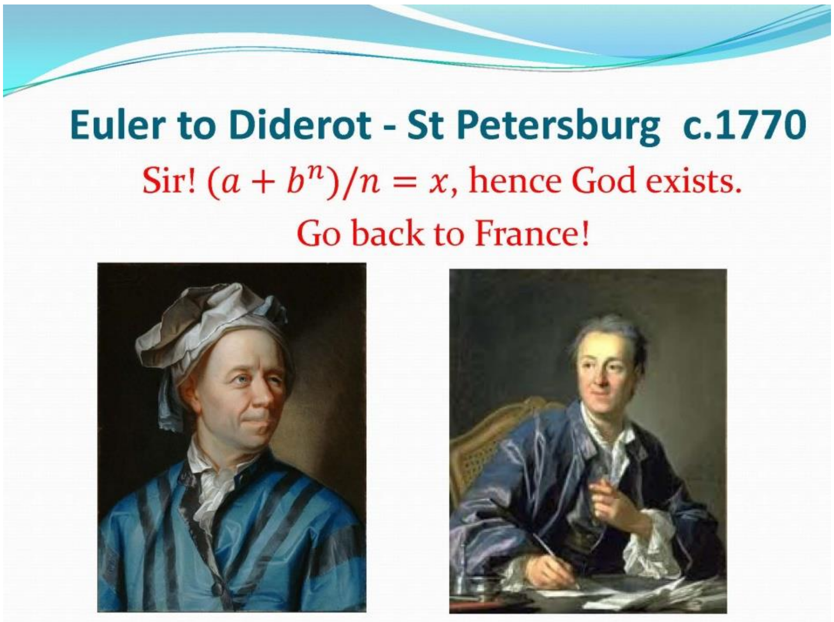

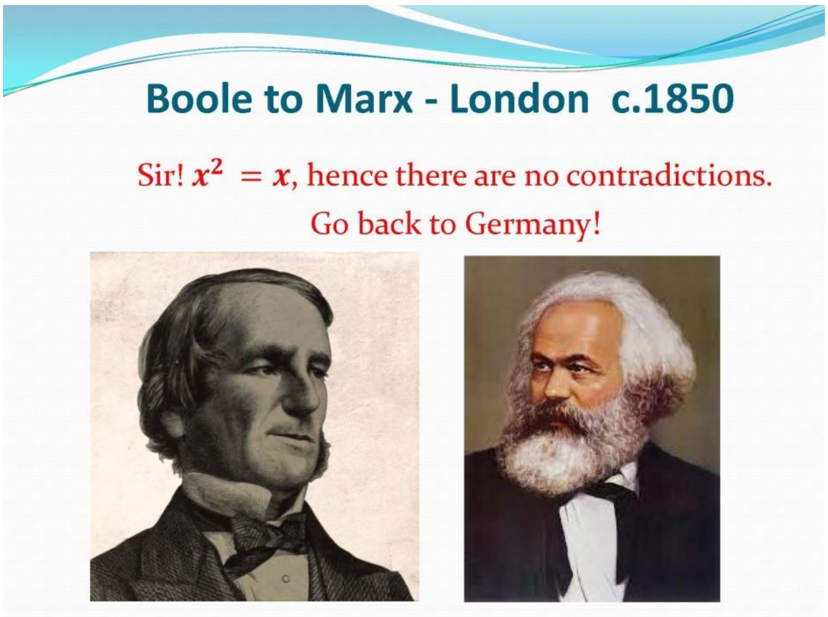

Is the principle of contradiction a consequence of $x^2 = x$?

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> *Many people think that it is impossible to make algebra about anything except number. This is a complete mistake … The use of algebra is to free people from bondage.*
>
> Mary Everest Boole[1]
> Philosophy and Fun of Algebra

Abstract

According to Boole it is possible to deduce the principle of contradiction from what he calls the fundamental law of thought and expresses as $x^2 = x$. We examine in which framework this makes sense and up to which point it depends on notation. This leads us to make various comments on the development of modern logic.

Contents

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[1] Mary Everest Boole was the wife of George Boole and the niece of George Everest whose name was given to the Mount Everest. She had many interests in particular for mathematical pedagogy. She contributed to the preparation of *The laws of thought* and also to the early death of her husband.

1. A strange proposition

PROPOSITION IV of Chapter III of the *Laws of Thought* of Boole states the following:

> … the principle of contradiction … is a consequence of the fundamental law of thought, whose expression is $x^2 = x$

This is strange for two main reasons. Firstly, before Boole nobody considered that $x^2 = x$ was a fundamental law of thought. Secondly, it is not clear how we can derive the principle of contradiction from this fundamental law. In this paper we will discuss only this second aspect. But this is related to the first point, because if we can derive a principle as fundamental as the principle of contradiction from $x^2 = x$, this is a good reason to consider that $x^2 = x$ is a fundamental law.[2]

This proposition can be seen as establishing connections between two heterogeneous fields, on the one hand metaphysics, on the other hand mathematics. We can draw a parallel with the famous controversy in St Petersburg in the 1770s when Leonard Euler told Denis Diderot:

Sir! $(a + b^n)/n = x$, therefore God exists, respond!

Since algebra sounded for Diderot like Hebrew he was ridiculed and ran back to Paris with the tails between its legs.[3]

A remake of this story could be as follow: during a meeting in the 1850s in London, George Boole told Karl Marx

Sir! $x^2 = x$, therefore there are no contradictions, respond!

Since algebra sounded for Marx like Arabic he was ridiculed and ran back to Germany with the tails between its legs.[4]

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[2] Note that Boole did not write : the fundamental law of thought, whose expression is "$x^2 = x$". What we are talking about is the proposition / thought corresponding to "$x^2 = x$", which can also alternatively denoted by "$xx = x$" or "$x.x = x$" or "$x \times x = x$" depending on the sign we are using for multiplication.

[3] This story was reported by different people, so there are slightly different versions, see (Gillings 1954). Funny enough the name "Denis Diderot" was given to the University Paris 7 known for having one of the best department of mathematics in France (including the strongest group of logicians in France and also of categoricists – This is where the author of this paper did a Master (on paraconsistent logic) and a PhD (on universal logic). The reason to choose Diderot's name was because the University of Paris 7 is the most pluridisciplinary university of Paris, this fits well with the fact that Diderot was an Encyclopedist. After May 1968, the University of Paris was split into many universities, each having a specialty and a number. Paris 7 was created in 1971 and the name "Denis Diderot" officially given to it in 1994. Since 2014 it is part of *Sorbonne Paris Cité* an administrative organization developed to re-unify the main universities of Paris.

[4] As it is known Marx moved to London in 1849. Boole was living at this time in Ireland. But we may imagine a meeting organized by Queen Victoria in London with both of them and other great intellectuals such as Stanley Jevons, Lewis Carroll, Friedrich Engels (who joined Marx in England in 1849). Marx and Engels were quite weak in mathematics although the latter tried to use mathematics to support their politico-philosophical theories. The bad quality of these writings was one of the reasons why van Heijenoort abandoned Marxism after having been 10 years secretary and bodyguard of Trotsky in Mexico (cf. Feferman 1993, 2012) –see his 1948 essay on Engels and mathematics. JvH had stopped his studies of mathematics in Paris when he was young, to join Trotsky. He was chosen because of his knowledge of many languages, knowledge he later used to work in the history of modern logic. JvH has promoted the myth of Frege as the founder of modern logic (cf his book *From Frege to Gödel* and his paper "Historical development of modern logic"). He had no interest for Boole. According to his PhD student I.Anellis, this is due in particular for a personal conflict he had at Brandeis University with one of his colleagues (see Anellis 1994).

If we put the above proof of PROPOSITION IV of the *Laws of Thought* as an exercise for a student, or even a professor, of the University of Foxbridge (S)he will probably not be able to present such a proof. (S)he may even claim that there is no such a proof because it is false.

The examination of the proof of PROPOSITION IV is interesting for various reasons. It is related to the question of notation. Does this proof depend on the concept of notation? Does any proof depend on notation? Of course everybody knows that it is not the same to compute using Roman numerals or Indo-Arabic numerals, but on the one hand computing is not the same as proving (even if we consider that proofs are recursive), and on the other hand the result, independently of the notation, is the same.

And what is the relation between notation and conceptual framework? Two notations may differ only due to superficial aspects. We can change the fonts of our text, choosing bigger fonts or another type of fonts. We can also transcribe a Russian text using Latin alphabet; this does not mean we are translating Russian into Latin.

But sometimes change of notation goes hand to hand with change of conceptual framework. A pictogramatic language does not work in the same way as an alphabetic language – we can still however claim that a Chinese basically thinks in the same way as a Russian.

When we go to mathematical notation, things are not the same. It is much more than a change of language; it is a change of way of thinking.[5]

Using mathematics do deal with logic, Boole changed the theory of reasoning. He also changed the way of reasoning and changed mathematics, introducing non numerical algebra, going out of the sphere of quantities. In his 1898 book, *A Treatise on Universal Algebra*, Whitehead wrote the following (p.35): "The Algebra of Symbolic Logic is the only known member of the non-numerical genus of Universal Algebra", with the footnote: "This algebra in all essential particulars was invented by Boole, cf. his work entitled, *An Investigation of the Laws of* Thought, London, 1854.".

As it is known Boole did not start from nothing, his work was developed under the influence of the British school of symbolic algebra promoted by George Peacock (1791-1858) and Duncan Farquharson Gregory (1812-1844) – about this school, see in particular the works of Marie-José Durand-Richard, also (Grattan Guinness 2000) and (Peckhaus 2009).[6]

2. Analysis of the original three-step derivation of Boole

The center of our attention is the following proof in three steps:

(1B)	$x^2 = x$
(2B)	$x - x^2 = 0$
(3B)	$x(1-x) = 0$

TABLE 1 – BOOLE ORIGINAL DERIVATION

[5] (Serfati, 2005) is a very interesting book about the development of mathematical symbolism.

[6] We will not discuss here the relation between Boole and Leibniz, in particular the distinction between *Lingua Universalis and Calculus Ratiocinator* (Couturat 1904 is still one of the best presentations of this distinction in Leibniz), nor will we discuss the relation between Boole and computation.

Each line of the proof is as in the original work of Boole. The three equations on the right column are written exactly in this way in the first edition of *The Laws of Thought* on page 49 (III.15).

The original version of Boole's *Laws of thought* has been reproduced in the Dover edition with the same pagination. A scan of the original book is available on the internet by Internet Archive, a non-profit digital library founded by Brewster Khale. Note also that Boole uses Roman numbers for Chapter and to each paragraph is attributed an Indo-Arabic number. So it is quite easy to make precise references to the Bible of modern logic.

Boole does not present this in a table and does not use the word "proof" to qualify what we call here a "3-step derivation". We are using a table to make easier the reading and to compare with other derivations that we will present, compare and discuss.

The word "derivation" means here *a development of a theorem*. This is item 5a of the definition of "derivation" in Dictionary.com It is commonly used in this sense both by specialists of proof-theory and by non-specialists, lay mathematicians. It is more neutral than "proof" and leaves open the degree of analysis and formalism.

This is why we will use this word and, as it is also commonly done, we can use it both to speak of the whole process, or to speak of a specific part of it: going from step 1 to step 2 is a derivation.

Boole successively presents these three equations in the order presented in TABLE 1. (1B) is presented at the end of PROPOSITION IV and he does not repeat it in a separate line. (2B) is presented at a middle of a line ending with a comma:

$$x - x^2 = 0,$$

and so is (3B) but with an semicolon and "(1)" at the very end on the right side:

$$x\,(1 - x) = 0; \qquad (1)$$

To write these three equations successively in separate lines to express the fact that we go from the first to the third through the second and that all are true is a procedure which still common in contemporary books of algebra as well as the details of the fonts: italic for the variable, no italics for the numbers, the parentheses, the exponent and the identity sign. It is also common to write something like "(1)" at the beginning or the end of the line, using it as a label for the asserted equation. Centering the equation is also common as well as the use of punctuation marks. The whole thing is like a sentence, the idea to divide the sentence starting new lines is to emphasize the assertion of each equation, their truth.

As it is known Frege introduced the sign "⊢" to make the distinction between a proposition and its assertion (cf Frege 1879; for an analysis of it, see Smith, 2009) . Before him this distinction was operated by this "lining-device". Frege's sign what adopted by Whitehead and Russell in *Principia Mathematica* (1910-1913) but not by Hilbert who didn't like it and kept using the traditional lining-device (cf Hilbert-Ackerman 1928).

This way of writing (lining-device, italic/non-italic fonts, label) derivations of equations was developed before Boole during the 18th century. See below an extract of Cauchy's famous *Cours sur le Calcul infinitésimal* (1823):

COURS D'ANALYSE. 13

QUATRIÈME LEÇON.

Différentielles des Fonctions d'une seule variable.

SOIENT toujours $y=f(x)$ une fonction de la variable indépendante x, i une quantité infiniment petite, et h une quantité finie. Si l'on pose $i=\alpha h$, α sera encore une quantité infiniment petite, et l'on aura identiquement

$$\frac{f(x+i)-f(x)}{i}=\frac{f(x+\alpha h)-f(x)}{\alpha h},$$

d'où l'on conclura

$$(1)\qquad \frac{f(x+\alpha h)-f(x)}{\alpha}=\frac{f(x+i)-f(x)}{i}\,h.$$

This way of writing has been preserved up to now independently of the development of mathematical logic (showing the weak, not say null, impact on logic upon mathematics, that for many people turns things more complicated without any further advantage).

In a standard contemporary book of algebra it is not necessarily explained what justifies the passage to the second equation from the first and to the third from the second. Students generally learn this as kind of informal rules.

Boole goes from step 1 to step 2, saying “let us write this equation in the form” and then presenting (2B). But after going form step 2 to step 3 on the basis of a “whence”, he qualifies both derivations by using the word “transformation”, saying “these transformations being justified by the axiomatic laws of combination and transposition (II.13)”.

We will not discuss here up to which point Boole was the first to be so precise. But all his work shows a careful analysis of the way mathematicians are proceeding. It would be interesting also to investigate the relation between the procedure describes by Boole and the development of the theory of “Rewriting” (about this theory, see e.g. N.Dershowitz and D.Plaisted, 2001).

3. Boolean algebra

If we present this 3-step derivation to a sample student, let say Natasha, she will think that it is part of algebra, not part of logic; in particular not part of propositional logic where there are not equations, no identity sign, except if we consider idiosyncratic constructions by someone like Roman Suszko (see e.g. Bloom and Suszko, 1972).

Which algebra? At first sight Natasha may think that it is about an algebra with 1 and 0 and other numbers on which x is ranging: natural numbers, rational numbers or reals numbers.

But looking more closely Natasha will change her mind because $x^2 = x$ is false if we consider that this free way to use the variable x corresponds to universal quantification, which is the standard interpretation.

What is also standard for a lay mathematician is to consider that the line-device corresponds to truth. So Natasha will only think of this derivation in the context where $x^2 = x$ is true. She will not consider the validity of the derivation independently of the truth of $x^2 = x$, only logicians do that or postmodern mathematicians.

If we consider that x is ranging over natural numbers $x^2 = x$ is generally false, just consider that the value of x is Natasha's age (for the sake of privacy we will not reveal it here). This equation is true among natural numbers exactly for two of them: 0 and 1. That is why Natasha will claim "This is (a derivation of) Boolean algebra".

If her teacher is a bit tricky she will correct her, adding: "Boolean algebra on {0,1} !". We now know that the Boolean algebra on {0,1} is the simplest Boolean algebra, but that a Boolean algebra can have a domain of more than two elements and not necessarily numbers: the power set of any set forms a Boolean algebra.

Let us emphasize that Boole's 3-step derivation can be performed/written exactly in the same way nowadays in standard mathematics. But people have totally forgotten the meaning originally given by Boole to it: derivation of the principle of contradiction from the fundamental law of thought. They generally don't interpret $x^2 = x$ as a fundamental law of thought and $x\,(1-x) = 0$ as the principle of contradiction.

About the relation between what is now called "Boolean algebra" and Boole's orginal work, see Hailpern 1981 and Burris 2000. The paper of Hailpern is symptomatically called "Boole's algebra isn't Boolean algebra". But considering the following quotation of Boole, it is reasonable to argue that he conceived what we now called the Boolean algebra on {0,1}:

> We have seen (II. 9) that the symbols of Logic are subject to the special law,
> $$x^2 = x$$
> Now of the symbols of Number there are but two, viz. 0 and 1, which are subject to the same formal law. We know that $0^2 = 0$, and that $1^2 = 1$; and the equation $x^2 = x$, considered as algebraic, has no other roots than 0 and 1. Hence, instead of determining the measure of formal agreement of the symbols of Logic with those of Number generally, it is more immediately suggested to us to compare them with symbols of quantity *admitting only of the values* 0 *and* 1. Let us conceive, then, of an Algebra in which the symbols *x, y, z*, etc. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. Upon this principle the method of the following work is established. II.15 (pp.26-27)

4. Boolean algebra from the point of view of first-order logic

Boolean algebra can be considered today from the point of view of classical first-order logic. First-order logic can itself be considered from two perspectives: model theory and proof theory. Sometimes this distinction is presented with a linguistic flavor as a distinction between semantics and syntax. This is rather ambiguous.

If we develop a strong link between syntax and computation, then many things that are called semantics can be called syntax. This is in fact what Chang and Keisler do in their famous book on model theory (1973) qualifying truth-tables as syntax. Moreover they define model theory with the following equation:

$$\textit{universal algebra} + \textit{logic} = \textit{model theory}$$

Although we can enjoy their sense of humor, structures do not reduce to algebras.[7] Model theory is a relation between mathematical structures and syntax, syntax not in the sense of proof-theory but in the sense of formulas and formulations. The study of syntax in this sense has been developed in details by logicians. Mathematicians are rather fuzzy about that.

Garrett Birkhoff's HSP theorem (1935), one of the main theorems in the pre-history of model theory (officially developed by Tarski, 1954-55), states that a class of algebras is closed under homomorphism, subalgebra and product iff it can be defined by a set of equations. It establishes a relation between operations on structures and syntactic expressions. We can re-formulate Chang and Keisler's equation as follows:

$$\textit{mathematical structures} \ \mathscr{H} \ \textit{syntax expressibility} = \textit{model theory}$$

We did not use the sign of addition but a sign conveying interaction.

The syntax expressibility of first-order logic is a sophisticated methodology that took many years to be developed (Cf. Hodges 1985-86). Here is how we can present Boole's derivation from this point of view:

(1FOL)	$\forall x \quad x \times x = x$
(2FOL)	$\forall x \quad x - (x \times x) = \mathbb{0}$
(3FOL)	$\forall x \quad x \times (\mathbb{1} - x) = \mathbb{0}$

TABLE 2 – FIRST-ORDER-LOGIC

The question is not to eliminate abbreviation for multiplication but to consider that multiplication and subtraction are not part of the syntax, nor are "0" and "1", the numerals for the numbers zero and one.

"$\times$" can be interpreted in different ways. Its meaning is fixed by some axioms. A sign similar to "x", a standard symbol for multiplication, is used, because it is the *intended-interpretation*, the same for "$\mathbb{1}$" and "$\mathbb{0}$". Without using the intended-interpretation methodology, the formula (3F) should be written as follows:

$$\forall x \quad x \, \mathbb{R} \, (\mathbb{a} \, \mathbb{S} \, x) = \mathbb{b}$$

$\mathbb{R}$ and $\mathbb{S}$ are signs for binary functions subject to various interpretations, $\mathbb{a}$ and $\mathbb{b}$ being signs for objects subject also to various interpretations.

The question of identity is also tricky; we will not discuss it here (see e.g. Beziau 1996). We could have put "=" and not "$=$", then identity would have been considered as a primitive notion, not subject to a variety of interpretations.

All what we have said here is much related with model-theory of first logic, "intepretations" refers to models. But this can also be viewed from the perspective of proof-

[7] In some sense, algebras are prototypes of mathematical structures. At some point the word "structure" was used as a name for lattice (cf Glivenko 1938). As it is known Bourbaki put the notion of structures at the heart of mathematics. They considered three basic mother structures: algebraic, topological and order structures (see Bourbaki 1948). About the development of abstract algebra and mathematical structures, see Corry 2004.

theory where "interpretation" is interpreted as the meaning of the operations and objects given by the rules of a proof-theoretical system.

Now than we have put the correct syntax, can we say that the three lines of TABLE 2 form a correct derivation? We can consider this derivation from first-order proof theory or from first-order model. In both cases the derivation is not true from a pure logical point of view (meaning of the quantifier and identity), we need to put some further axioms governing the non-logical entities.

Let us consider the derivation from a model-theoretical point of view. According to the notion of semantical consequence promoted by Tarski (1936)[8], (2FOL) follows from (1FOL) iff all models of (1FOL) are models of (2FOL) and (3FOL) follows from (2FOL) iff all models of (2FOL) are models of (3FOL), and going directly to the first line to the third line: (3FOL) follows from (1FOL) iff all models of (1FOL) are models of (3FOL).

This is no true for all models, but it is true for a certain class of models: rings. Ring is a sufficient condition, but not a necessary condition. And ring itself is not the correct terminology because a ring is generally defined with multiplication and addition, not with substraction. A structure can be defined in various equivalent ways. Stone (1935) has proven that an idempotent ring is equivalent to a complemented distributive lattice. An *idempotent ring* is a ring with as additional axiom the formula (1FOL), it also called a *Boolean ring*.

In model theory two structures are said to be equivalent iff they have a common expansion by definition up to isomorphism. The notion of expansion is a notion mixing syntax and semantics. An open question is: how can we conceive a structure independently of a given language, of a given signature?

Instead of interpreting Boole's three step derivation from the point of view of a class of models. We can consider it within a particular model, i.e. within the Boolean algebra on {0,1}:

(1BOL)	$b \times b = b$
(2BOL)	$b - (b \times b) = 0$
(3BOL)	$b \times (1 - b) = 0$

TABLE 3 – BOOLEAN ALGEBRA ON {0,1}

We have changed the name of the variable, renaming "x", "b", to emphasize that its domain of variation is {0,1}. We also have used back usual signs for multiplication, subtraction, identity, zero and one, because here the derivation is performed in the context of concrete operations and objects. In this case each line of the derivation can be justified as true by considering the two possible instantiations of the variable.

Let us note that in mathematics it is common to use the same sign, for example "0" to denote different objects, e.g. zero as a natural number, zero as an integer. This can be justified by the fact that one is an element of a substructure of the other structure. This phenomenon is indeed explicitly explained by model theory which justifies this "abus de langage".

[8] J.Corcoran and J.M. Miguel Sagüillob (2011) have argued that this notion was not really presented by Tarski in his 1936's paper but only during his Berkeley period (Tarski 1954-55).

5. Algebra of sets

We can say that the most famous *model* of Boolean algebra is the Boolean algebra on {0,1}. Other famous *models* are algebras of sets. It is possible to prove that the power set of any set forms a Boolean algebra and there is the famous Stone representation theorem (1936) stating that every Boolean algebra is isomorphic to a field of sets.

Let us rewrite the formulas of TABLE 2 with this intended interpretation. We have then the following table:

(1FAS)	$\forall x$	$x \Cap x \equiv x$
(2FAS)	$\forall x$	$x \setminus (x \Cap x) \equiv \varnothing$
(3FAS)	$\forall x$	$x \Cap (\mathbb{U} \setminus x) \equiv \varnothing$

TABLE 4 – FIRST-ORDER ALGEBRA OF SETS

The intended interpretation of $\Cap$, $\setminus$, $\varnothing$, $\mathbb{U}$ are respectively intersection, subtraction, the empty set and the universal set i.e. the set of all objects. We can rewrite TABLE 4 as follows.

(1AS)	$A \cap A = A$
(2AS)	$A \setminus A \cap A = \varnothing$
(3AS)	$A \cap (\boldsymbol{U} \setminus A) = \varnothing$
(4AS)	$A \cap \bar{A} = \varnothing$

TABLE 5 – INFORMAL ALGEBRA OF SETS

The derivation from (1AS) to (3AS) is generally performed using intuitive properties of set-theoretical operations, like when doing derivations about equations on numbers. This kind of derivation can be interpreted as "working in the model".

We have added a fourth line introducing the standard notion of set complementation, that can be considered as an abbreviation or a primitive notion.

$A \cap \bar{A} = \varnothing$ seems a reasonable formulation of the principle of contradiction. This can be supported by the confusion between contrariety and contradiction (see Beziau 2016). But if we are using the word contradiction more precisely as defined by the theory of the square of opposition, the following axiom should be added $A \cup \bar{A} = \boldsymbol{U}$. For example a square cannot be a circle, the intersection between the sets of square and circles ie empty, but something can be neither a circle nor a square, that's why these two notion are not contradictory. A better example is straight line and curve; contradictory concepts because obeying the additional axiom (see Beziau 2015). But what Boole calls the principle of contradiction is not the fourth line, but the third line, this avoids the confusion. Considering that $\bar{A} = \boldsymbol{U} \setminus A$, we indeed have $A \cup \bar{A} = \boldsymbol{U}$.

Boole had clearly in view this algebra of sets, but he was using the word "class" rather than "set" (Marx also used at this time this word; "Klasse" in German). He wrote;

> If x represent any class of objects, then will $1 - x$ represent the contrary or supplementary class of objects, i.e. the class including all objects which are not comprehended in the class x. For greater distinctness of conception let x represent the class men, and let us express, according to the last Proposition, the

> Universe by 1; now if from the conception of the Universe, as consisting of "men" and "not-men," we exclude the conception of "men," the resulting conception is that of the contrary class, "not-men." Hence the class "not-men" will be represented by $1-x$. And, in general, whatever class of objects is represented by the symbol , the contrary class will be expressed by $1-x$. III.14

Boole identifies (3AS) with (4AS) and (4AS) with the principle of contradiction as stated by Aristotle (which is in fact in view of the theory of oppositions originated by Stagirite himself a confusion between contradiction and incompatility):

> the equation (1) $x\,(1-x)\,=0$ thus express the principle, that a class whose members are at the same time men and not men does not exist. In other words, that it is impossible for the same individual to be at the same time a man and not a man. Now let the meaning of the symbol x be extended from the representing of "men," to that of any class of beings characterized by the possession of any quality whatever; and the equation (1) will then express that it is impossible for a being to possess a quality and not to possess that quality at the same time. But this is identically that "principle of contradiction" which Aristotle has described as the fundamental axiom of all philosophy. "It is impossible that the same quality should both belong and not belong to the same thing.. . . This is the most certain of all principles.. Wherefore they who demonstrate refer to this as an ultimate opinion. For it is by nature the source of all the other axioms." (Metaphysics III; 3). III.15

Couturat in his famous booklet *L'algèbre de la logique* published in France in 1905 and translated few years later (1914) in English presented a system with two interpretations, conceptual and propositional interpretations, and made the following comments:

> the algebra in question, like logic, is susceptible of two distinct interpretations, the parallelism between them being almost perfect, according as the letters represent concepts or propositions. Doubtless we can, with Boole and Schröder, reduce the two interpretations to one, by considering the concepts on the one hand and the propositions on the other as corresponding to assemblages or classes; since a concept determines the class of objects to which it is applied (and which in logic is called its extension), and a proposition determines the class of the instances or moments of time in which it is true (and which by analogy can also be called its extension). Accordingly the calculus of concepts and the calculus of propositions become reduced to but one, the calculus of classes, or, as Leibniz called it, the theory of the whole and part, of that which contains and that which is contained. But as a matter of fact, the calculus of concepts and the calculus of propositions present certain differences, as we shall see, which prevent their complete identification from the formal point of view and consequently their reduction to a single calculus of classes. Accordingly we have in reality three distinct calculi, or, in the part common to all, three different interpretations of the same calculus. In any case the reader must not forget that the logical value and the deductive sequence of the formulas does not in the

least depend upon the interpretations which may be given them, and, in order to make this necessary abstraction easier, we shall take care to place the symbols C. I.(conceptual interpretation) and P. I. (propositional interpretation) before all interpretative phrases. These interpretations shall serve only to render the formulas intelligible, to give them clearness and to make their meaning at once obvious, but never to justify them. They may be omitted without destroying the logical rigidity of the system. (Couturat 1914, p.2)

In the next section, we will see how the propositional interpretation can be seen today from the point of view of the theory of models.

6. Classical propositional logic

Let us finally examine the relation between Boole's original derivation and classical propositional logic (hereafter CPL), not CPL as conceived by Boole, but as it is nowadays understood.

It is common to claim that CPL is a Boolean algebra. What does it mean exactly? Model theory permits a clear formulation of the problem: can CPL be considered as a first-order structure which is a model of the axioms of a Boolean ring or an equivalent structure ?

Having in mind CPL as intended-interpretation we can rewrite TABLE 2 as follows:

(1P)	$\forall x$	$x \wedge x \dashv\vdash x$
(2P)	$\forall x$	$x - (x \wedge x) \dashv\vdash \perp$
(3P)	$\forall x$	$x \wedge (\top - x) \dashv\vdash \perp$

TABLE 6 – FIRST-ORDER-CLASSICAL PROPOSITIONAL LOGIC

As we have simplified TABLE 4 into TABLE 5, we can simplify TABLE 6 into TABLE 7 :

(1P)	$p \wedge p \dashv\vdash p$
(2P)	$p - (p \wedge p) \dashv\vdash \perp$
(3P)	$p \wedge (\top - p) \dashv\vdash \perp$
(4P)	$p \wedge \neg p \dashv\vdash \perp$

TABLE 7 – CLASSICAL PROPOSITIONAL LOGIC

Similarly as in TABLE 5 we have introduced a 4th line In TABLE 5, we used "A" as a sign for set, in table 5 "p" as a sign for proposition, similarly as "n" is generally used as a sign for natural number. In TABLE 2, TABLE 4 and TABLE 6 we have used "x" as a sign for variable, more precisely a *bound variable*. In TABLE 1 "x" is used as a free variable, there are not quantifiers. As it is known quantifiers were introduced only after Boole (By Peirce, Schröder, Frege, the symbol "$\forall$" itself having been introduced by Gentzen). The letter "x" as a free variable was introduced by Descartes. It is common in CPL to talk about "propositional variable", but it is not necessarily clear what is meant by this expression. Sometines "p" is used only as a sign for atomic proposition. Here we are using it as sign for any proposition, atomic, or molecular, in other words, for any formula.

An important difference between TABLE 7 and TABLES 3 and 5 is that ⊣⊢ is not identity. The symbol " ⊣⊢ " is not so much used in logical books. "$p \wedge p \dashv\vdash p$" can be seen as an abbreviation of "$p \wedge p \vdash p$ and $p \vdash p \wedge p$". The relation denoted by " ⊣⊢ " is called *logical equivalence*. In case of CPL it is possible to show that it is a congruence relation and then we can consider the factor structure which is called a *Tarski-Lindenbaum* algebra or sometimes just a *Lindenbaum* algebra. This construction can be generalized to any logical structure where logical equivalence is a congruence relation. These logics are called *auto-extensional* In the factor structure ⊣⊢ is then identity. What can be said is that, in case of CPL, the Tarski-Lindenbaum is a Boolean algebra.[9]

Now can we say that TABLE 7 is about classical propositional logic? There are different ways to present CPL, not necessarily equivalent (see Beziau 2001). In the Polish tradition it is common to consider a propositional logic as a structural consequence relation. This is what we are doing here, using the sign "⊢"to denote the consequence relation, i.e. a relation between theories (sets of formulas) and formulas.

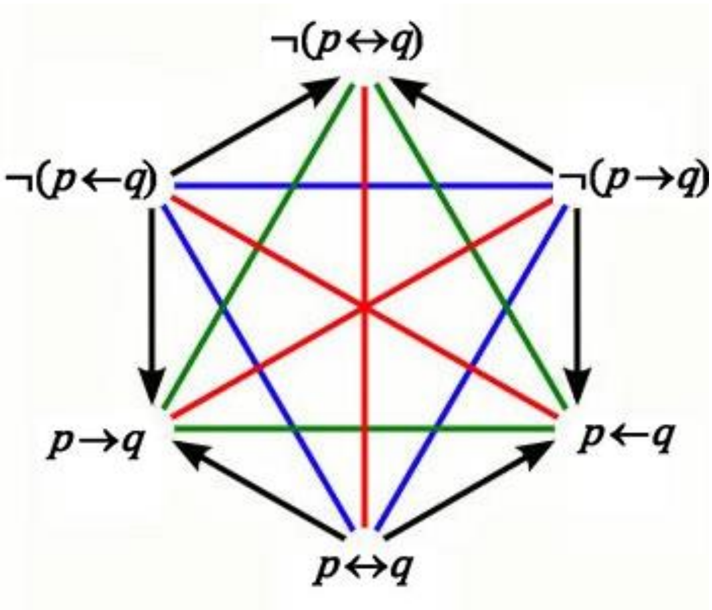
In TABLE 7 we have ⊤ and ⊥. They are not always considered in CPL but their use is known. The origin of the sign "⊤" is probably due to the fact that "⊤" is the first letter of "True". However the constant "⊤" does not denote truth but logical truth, it is something which is always true: $\vdash \top$. The sign "⊥" is not the first letter of "False", it was probably chosen to graphical express the contrast between truth and falsity. However "⊥" does not denote falsity but logical falsity. It is something which is always false, this can be expressed by: $\bot \vdash p$. ⊤ and ⊥ can also be used in other logics, than CPL, they do not characterized CPL. At some point Tarski considered that the presence of ⊥ was an axiom for any logic (see Beziau 2006). It would be completely confuse here to use "1" and "0" instead of "⊤" and "⊥", since nowadays "1" and "0" are used to denote truth-values which are not part of the syntax of CPL.

What about "—"? It is rarely used in CPL. It is called "logic subtraction" and has been used in particular by Haskell Curry (1952). Let us consider the following four connectives:

p	q	$p \longrightarrow q$	$\neg(p \longrightarrow q)$	$p \longleftarrow q$	$\neg(p \longleftarrow q)$
0	0	1	0	1	0
0	1	1	0	0	1
1	0	0	1	1	0
1	1	1	0	1	0

The relations between them can be pictorially described by Blanché's hexagon of opposition (see Blanché 1957, Béziau 2012, Wybraniec-Skardowska 2016):

[9] This construction was performed for CPL by Tarski (1935), not exactly in this way, he was considering the connective of bi-implication, not the relation of logical equivalence. But we can say that Tarski proved that CPL is a Boolean algebra. The first step is to consider that connectives from a syntactic point of view are functions and that that the set of formulas is an algebra, an absolutely free algebra (idea attributed to Lindenbaum).



The following table shows that if we want to define $\neg p$ as $\top - p$ we have to consider that subtraction corresponds to $\neg(p \longleftarrow q)$:

p	$\top$	$p \longrightarrow \top$	$\neg(p \longrightarrow \top)$	$p \longleftarrow \top$	$\neg(p \longleftarrow \top)$	$\neg p$
0	1	1	0	0	1	1
1	1	1	0	1	0	0

We can alternatively use the sign "$\nleftarrow$" for subtraction and "$\nrightarrow$", as a sign for the negation of implication. We know that the following 18 sets of two connectives are truth-functionally complete:

$\vee$ $\neg$	$\rightarrow$ $\nleftrightarrow$	$\leftrightarrow$ $\nrightarrow$
$\wedge$ $\neg$	$\leftarrow$ $\nleftrightarrow$	$\leftrightarrow$ $\nleftarrow$
$\rightarrow$ $\neg$	$\rightarrow$ $\bot$	$\rightarrow$ $\nrightarrow$
$\leftarrow$ $\neg$	$\leftarrow$ $\bot$	$\leftarrow$ $\nrightarrow$
$\nrightarrow$ $\neg$	$\nrightarrow$ $\top$	$\rightarrow$ $\nleftarrow$
$\nleftarrow$ $\neg$	$\nleftarrow$ $\top$	$\leftarrow$ $\nleftarrow$

So if we interpret the connectives of TABLE 7 classically we can say that this is a derivation in CPL. We may also want to perform this derivation in a subsystem of classical logic, in particular with a weaker negation.

7. Formalization of the principle of non-contradiction - PNC

Can we consider that

$$p \wedge \neg p \ \dashv\vdash \bot$$

is a good formulation of the principle of non-contradiction?

From the right to the left it is a consequence of the fundamental axiom for $\bot$, known in Latin as *ex falso sequitur quo libet*, that can be written in symbols as follows:

$$\bot \ \vdash q$$

So let us examine the other direction

$$p \wedge \neg p \ \vdash \bot$$

Considering the ex-falso, this can be rewritten

$$p \wedge \neg p \ \vdash \ q$$

This sometimes is called ex *contradictione sequitur quod libet*. But is $p \wedge \neg p$ really a contradiction?

First let us note that $p, \neg p \vdash \ q$ is not equivalent to $\vdash \neg(p \wedge \neg p)$. The two are independent as it can be shown by the following table:

p	$\neg p$	$p \wedge \neg p$	$\neg(p \wedge \neg p)$
0	1	0	1
½	½	½	½
1	0	0	1

This is a table using a three-valued logical matrix. If ½ is considered as designated, $\vdash \neg(p \wedge \neg p)$ is valid but not $p, \neg p \vdash \ q$. If ½ is considered as non designated, $p, \neg p \vdash \ q$ is valid but not $\vdash \neg(p \wedge \neg p)$.

The standard definition of paraconsistent logic is based on the rejection of $p, \neg p \vdash \ q$ (see e.g. Beziau 2000) but there are paraconsistent logics in which also $\vdash \neg(p \wedge \neg p)$ is not valid. We have recently called such logics, g*enuine paraconsistent logics* and developed two three-valued genuine paraconsistent logics (see Beziau 2016).

In classical logic, we have the validity of both formulations of the principle of contradiction, but can we consider that the two together forms the principle of contradiction? Not in the sense of contradiction as in the square of opposition, because they are independent of the principle of excluded middle, this means they don't exclude contrariety. The situation is parallel as the one at the hand of our section 5. A formulation of the principle of contradiction that is compatible with the square definition would be

$$p \wedge (\top - p) \ \vdash \bot$$

and this is derivable from

$$p \wedge \ p \ \dashv\vdash p.$$

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