Chapter 7

MANY-VALUED AND KRIPKE SEMANTICS

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7.1 Introduction

Many-valued and Kripke semantics are generalizations of classical semantics in two different "opposite" ways. Many-valued semantics keep the idea of homomorphisms between the structure of the language and an algebra of truth-functions, but the domain of the algebra may have more than two values. Kripke semantics keep only two values but a relation between bivaluations is introduced.

Many-valued semantics were proposed by different people among whom Peirce, Łukasiewicz, Post, Bernays. In fact all these people are also considered as founders of the semantics of classical zero-order logic (propositional logic). And from their work it appears that the creation of many-valued semantics is almost simultaneous to the creation of the bivalent two-valued semantics. From this point of view we cannot say that many-valued semantics are an abstract meaningless generalization developed "après coup", as suggested by Quine in ([Quine 1973], p. 84). However it is true that the meaning of the "many" values is not clear. As Quine and other people have noticed, the division between distinguished and non distinguished values in the domain of the algebra of truth-functions of many-valued semantics is clearly a bivalent feature. So, in some sense many-valued semantics are bivalent, in fact they can be reduced, as shown for example by Suszko, to bivalent (non truth-functional) semantics. Suszko was also against the terminology "logical values" for these many values.

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He thought that Łukasiewicz was seriously mistaken to consider the third value of his logic as possibility (see [Suszko 1977] and also [da Costa, *et al.* 1996], [Tsuji 1998]). I don't share Suszko's criticism on this point. It seems to me that the many values can be conceived as degrees of truth and degrees of falsity and that we can consider a four-valued semantics in which the two distinguished values can be called "possibly true" and "necessary true", and the two non distinguished values can be called "possibly false" and "necessary false". With this intuition we can develop a four-valued modal logic [Dugundji 1940]. The use of many-valued semantics for the development of modal logic has been completely left out. This can be explained by two reasons: on the one hand the negative results proved by Dugundji showing that S5 and other standard modal logics cannot be characterized by finite matrices [Dugundji 1940], on the other hand the rise of popularity of Kripke semantics.

Today many people identify Kripke semantics with modal logic. Typically a book called "modal logic" nowadays is a book about Kripke semantics (cf. e.g. the recent book by [Blackburn *et al.* 2001]). But modal logic can be developed using other kinds of semantics and Kripke semantics can be used to deal with many different logics and it is totally absurd to call all of these logics "modal logics". Kripke semantics are also often called "possible worlds semantics", however this is quite misleading because the crucial feature of these semantics is not the concept of possible world but the relation of accessibility. Possible worlds can easily be eliminated from the definition of Kripke semantics and then the accessibility relation is defined directly between the bivaluations. For this reason it seems better to use the terminology "relational semantics". Of course, if we want, we can call these bivaluations "possible worlds", this metaphor can be useful, but then why using this metaphor only in the case of relational semantics? In fact in the *Tractatus* Wittgenstein used the expression "truth-possibilities" for the classical bivaluations. Other concepts of the semantics of classical zero-order logic were expressed by him using a modal terminology: he said that a formula is necessary if it holds for all truth-possibilities, impossible if it holds for none, and possible if it holds for some. But Wittgenstein was against the introduction of modal concepts inside the language as modal operators.

Many-valued and Kripke semantics may be philosophically controversial, anyway they are very useful and powerful technical tools which can be fruitfully used to give a mathematical account of basic philosophical notions, such as modalities. It seems to me that instead of focusing on the one hand on some little philosophical problems and on the other hand on some developments limited to one technique, one should promote a better interaction between philosophy and logic developing a wide range of techniques, as for example the combination of Kripke semantics (extended as to include the semantics Jaskowski) and Many-valued semantics (extended as to include non truth-functional many-valued semantics). My aim is this paper is to give a hint of how these techniques can be developed by presenting various examples.

7.2 Many-valuedness and modalities

Many people have nowadays forgotten that the first formal semantics for modal logic was based on many-valuedness. This was first proposed by Łukasiewicz in 1918 and published in [Łukasiewicz 1920]. Moreover many-valued logic was developed by Łukasiewicz in view of modalities, he introduced a third value which was supposed to represent possibility. Although there is no operator of possibility in the standard version of Łukasiewicz's three-valued logic $Ł_3$, at first there was one,, eliminated after Tarski showed that it was definable in terms of other non modal connectives.

Łukasiewicz's logic was dismissed as a modal logic by many people, since it has strange features like the validity of the formulas: $\diamond a \wedge \diamond b \rightarrow \diamond(a \wedge b)$. Later on, in 1940, the negative result of Dugundji showing that some of the famous Lewis's modal systems like S4 and S5 cannot be characterized by finite matrices was another drawback for the many-valued approach to modal logic. Nevertheless Łukasiewicz insisted in this direction and in 1953 he presented a four-valued system of modal logic [Łukasiewicz 1953]. This system is also full of strange features and was never taken seriously by modal logicians. At the end of the 1950s the rise of Kripke semantics put a final colon to the love story between many-valuedness and modalities. Nowadays the many-valuedness approach to modal logic is considered as prehistory.

However I think it is still possible to develop in a coherent and intuitive way many-valued systems of modal logic. A possible idea is to consider a set of four-values, two non distinguished values, 0^- and 0^+, and two distinguished values, 1^- and 1^+. These values are ordered by the following linear order: $0^- \prec 0^+ \prec 1^- \prec 1^+$. A possible interpretation is to say that 0^- means necessary false, 0^+ possibly false, 1^- means possibly true and 1^+ means necessary true.

The basic laws for modalities are the following:

$$\begin{array}{ll} \Box a \vdash a & a \vdash \diamond a \\ a \nvdash \Box a & \diamond a \nvdash a \\ \Box a \vdash \diamond a & \diamond a \nvdash \Box a \end{array}$$

In order for these laws to be valid the tables defining possibility and necessity *must* obey the conditions given by the following table:

a	$\Box a$	$\diamond a$
0^-	0	0
0^+	0	1
1^-	0	1
1^+	1	1

Table 1.

In this table 0 means 0^- or 0^+ and 1 means 1^- or 1^+.

We have many possibility Nevertheless all systems obeying the conditions given by Table 1 obey the involution laws:

$\Box a \dashv\vdash \Box\Box a$

$\diamond a \dashv\vdash \diamond \diamond a$

the De Morgan laws for modalities:

$\Box a \wedge \Box b \dashv\vdash \Box(a \wedge b)$

$\diamond a \vee \diamond b \dashv\vdash \diamond(a \vee b)$

as well as Kripke law, considering that implication is defined classically as $\neg a \vee b$ and that disjunction is standardly defined with the operator min:

$\Box(\neg a \vee b) \vdash \neg\Box a \vee \Box b.$

One possibility for the minus/plus choice is to reduce the four values to two values 0^- and 1^+. We get then the following table:

a	$\Box a$	$\diamond a$
0^-	0^-	0^-
0^+	0^-	1^+
1^-	0^-	1^+
1^+	1^+	1^+

Table 2. M4-Red

With this idea we get the collapse of compound modalities:

$\diamond a \dashv\vdash \Box \diamond a$

$\diamond\Box a \dashv\vdash \Box a$

We are getting therefore very close to S5, although we know, due to Dugundji's theorem that this table cannot define S5. So what are the laws of S5 which are not valid in M4-Red? It depends on the way that we define the non modal connectives. We can reduce the four values to two values 0^- and 1^+ for these connectives or not.

If we do not operate the reduction, we have the standard definitions for conjunction and disjunction with the operators min and max defined on the linear order, and we define the negation in the following logical way:

a	$\neg a$
0^-	1^+
0^+	1^-
1^-	0^+
1^+	0^-

Table 3.

In this case the rule of necessitation

if $\vdash a$ then $\vdash \Box a$

is not valid, as shown by the following table:

p	$\neg p$	$p \vee \neg p$	$\Box(p \vee \neg p)$
0^-	1^+	1^+	1^+
0^+	1^-	1^-	0^-
1^-	0^+	1^-	0^-
1^+	0^-	1^+	1^+

Table 4.

The fact that the rule of necessitation is not valid can be seen as a serious defect. However, Łukasiewicz has argued at length against the validity of such rule (see [Łukasiewicz 1954]).

Another possibility is to operate a reduction of two values for all molecular formulas. In this case, we get a logic in which the law of necessitation is valid but in which self-extensionality

if $a \dashv\vdash b$ then $\Box a \dashv\vdash \Box b$

if $a \dashv\vdash b$ then $\diamond a \dashv\vdash \diamond b$

does not hold.

7.3 Possible worlds semantics without possible worlds

It seems that possible worlds are, as stressed by the name, essential in possible worlds semantics.

In possible worlds semantics we have possible worlds and this would be the difference with classical semantics or many-valued semantics. So an expression like "possible worlds semantics without possible worlds" sounds a bit paradoxical like "orange juice without orange", etc. But in fact, as we will see, possible worlds can easily be eliminated from the standard definition leading to a definition which is equivalent in the sense that it defines the same logics.

There are several presentations of possible worlds semantics, let us take a standard one, close to the one given by Johan van Benthem (cf. [van Benthem 1983]).

We consider a Kripke structure $K =< W, R, V >$, as a set W of objects called *possible worlds*, a binary relation R between these worlds called *accessibility relation*, and a function V assigning a set of possible worlds to each atomic formula. Then we give the following definition:

DEFINITION PWS.

(0) $\models_w p$ iff $w \in V(p)$

(1) $\models_w \neg a$ iff $\not\models_w a$

(2) $\models_w a \wedge b$ iff $\models_w a$ and $\models_w b$

(3) $\models_w a \vee b$ iff $\models_w a$ or $\models_w b$

(4) $\models_w a \rightarrow b$ iff $\not\models_w a$ or $\models_w b$

(5) $\models_w \Box a$ iff for every $w' \in W$ such that wRw', $\models_{w'} a$

(6) $\models_w \diamond a$ iff for some $w' \in W$ such that wRw', $\models_{w'} a$

What does mean this definition? What does this definition define? It defines a binary relation between the worlds of W of K and formulas, badly expressed by the notation $\models_w a$. This can be read as "the formula a is true in the world w". From this definition, we then define what it means for a formula a to be true in the Kripke structure K: a is true in K iff it is true in every world w of K.

As we see, in these definitions, the nature of the worlds is never used, they can be anything. Why then calling them worlds? What is used is the relation of accessibility: different properties of this relation lead to different logics.

The second important point is that the definition defines a binary relation between the worlds of W of K and formulas by *simultaneous* recursion: in clauses (4) and (5), to define the relation between a world w and a formula, we use the relation defined between another world w' and formulas. In classical semantics and many-valued semantics, we only use *simple* recursion.

Let us now transform this definition into a worldless definition Instead of considering a Kripke structure, we consider a Ipke structure

$I =< D, R >$

as a set D of functions called *distributions of truth-values* assigning to every atomic formula a the values 0 (false) or 1 (true), and a binary relation R between these distributions called *accessibility relation.*

We now extend these distributions into bivaluations, i.e. function assigning to every formula (atomic or molecular) the values 0 (false) or 1 (true).

DEFINITION PWS-W.

(0) $\beta_\delta(p) = 1$ iff $\delta(p) = 1$

(1) $\beta_\delta\neg(a) = 1$ iff $\beta_\delta(a) = 0$

(2) $\beta_\delta(a \wedge b) = 1$ iff $\beta_\delta(a) = 1$ and $\beta_\delta(b) = 1$

(3) $\beta_\delta(a \vee b) = 1$ iff $\beta_\delta(a) = 1$ or $\beta_\delta(b) = 1$

(4) $\beta_\delta(a \rightarrow b) = 1$ iff $\beta_\delta(a) = 0$ or $\beta_\delta(b) = 1$

(5) $\beta_\delta(\Box a) = 1$ iff for every $\beta'_{\delta'}$ such that $\delta R \delta'$, $\beta'_{\delta'}(a) = 1$

(6) $\beta_\delta(\diamond a) = 1$ iff for some $\beta'_{\delta'}$ such that $\delta R \delta'$, $\beta'_{\delta'}(a) = 1$

Using the above definition, we can then define, what it means to be true in the Ipke structure I: a formula a is true iff it is true for every bivaluation.

EQUIVALENCE OF THE TWO DEFINITIONS. *It is the same to be true in a Kripke structure or to be true in an Ipke structure.*

This claim means more precisely that given a Kripke structure, we can construct an Ipke structure which leads to the same notion of truth and vice-versa. The construction is very simple. Given a Kripke structure, we transform a possible world w into a distribution δ_w by putting $\delta_w(p) = 1$ iff $w \in V(p)$. Given an Ipke structure, we transform a distribution δ into a possible world w_δ obeying the condition: $w \in V(p)$ iff $\delta(p) = 1$. This condition in fact defines the function V.

In both cases the accessibility relation is transposed from worlds to distributions and vice-versa.

Someone may claim that possible worlds are nice tools, they help imagination, they are *heuristical*. But we may call bivaluations in DEFINITION PWS-W, possible worlds. We still get the heuristics, but keep a low ontological cost. In fact some people even call possible worlds, the bivaluations of the standard semantics of classical propositional logic, following the first idea of Wittgenstein.

In some recents advances in possible worlds semantics (Dutch trend), possible worlds may be useful, but they are totally useless for the standard semantics of $S5$, etc. On the other hand to work without possible worlds can simplify further constructions as the ones presented in the next sections.

7.4 Combining many-valued and Kripke semantics

If we consider possible worlds semantics without possible worlds, i.e., given by DEFINITION PWS-W, it is easy to combine them with many-valued semantics: instead of considering bivaluations, we consider functions into a finite set of values divided into two sets, the sets of distinguished values and the set of non-distinguished values. We will call such combined semantics *Many-valued Kripke semantics*.

Sometimes people talk about impossible worlds or incomplete worlds (see e.g. the volume 38 (1997) of *Notre Dame Journal of Formal Logic*). An impossible world is a world in which a formula and its negation can both be true, an incomplete world is a world in which a formula and its negation can both be false. These impossible worlds (or incomplete worlds) semantics can be described more efficiently by Many-valued Kripke semantics.

Let us give an example of many-valued relational semantics, we consider a many-valued Ipke structure $MI =< D, R >$, where D is a set of distributions assigning to every atomic formula a, the values 0, $\frac{1}{2}$ or 1 and where R is a binary relation of accessibility between these distributions. We now extend these distributions into three valuations, i.e. function assigning to every formula (atomic or molecular) the values 0, $\frac{1}{2}$ or 1.

DEFINITION MI.

(0) $\theta_\delta(p) = \delta(p)$

(1) $\theta_\delta\neg(a) = 1$ iff $\theta_\delta(a) = 0$

(2) $\theta_\delta(a \wedge b) = min(\theta_\delta(a), \theta_\delta(b))$

(3) $\theta_\delta(a \vee b) = max(\theta_\delta(a), \theta_\delta(b))$

(4) $\theta_\delta(a \rightarrow b)$ is distinguished iff $\theta_\delta(a)$ is non distinguished or $\theta_\delta(b)$ is distinguished.

(5) $\theta_\delta(\Box a) = 1$ is distinguished iff for every $\theta'_{\delta'} \in W$ such that $\delta R \delta'$, $\theta'_{\delta'}(a)$ is distinguished.

(6) $\theta_\delta(\diamond a) = 1$ is distinguished iff for some $\theta'_{\delta'} \in W$ such that $\delta R \delta'$, $\theta'_{\delta'}(a)$ is distinguished.

At first this definition seems quite the same as DEFINITION PWS-W of the preceding section, but since we have a third value, things change. From clause (1), we deduce that

(2') $\theta_\delta\neg(a) = \frac{1}{2}$ iff $\theta_\delta(a) = \frac{1}{2}$.

If we consider that 1 is distinguished and the values 0 and $\frac{1}{2}$ are non-distinguished, then the principle of contradiction expressed by the formula $\neg(p \wedge \neg p)$ is not true in MI, provided we standardly define "true in MI" by "distinguished for every three-valuations": we have some three-valuations in which both values of p and $\neg p$ are $\frac{1}{2}$, and therefore in which the value of $\neg(p \wedge \neg p)$ is $\frac{1}{2}$, i.e. non-distinguished. This is nothing very knew and this is what happens in Łukasiewicz three-valued logic Ł3, where we have:

$$\nvdash \neg(a \wedge \neg a)$$

We are just *combining* different semantics. What happens here is that, at the level of modalities, we don't either have the principle of non contradiction:

$$\nvdash \neg(\Box a \wedge \neg \Box a) \qquad \nvdash \neg(\diamond a \wedge \neg \diamond a)$$

If we take $\frac{1}{2}$ and 1 as distinguished and only 0 as non-distinguished and provided we define the consequence relation in the usual way, then the formulas above are valid but the formulas below expressing the *ex-falso sequitur quod libet* which are valid with only 1 as distinguished are not valid anymore:

$$\begin{array}{ll} \nvdash (p \wedge \neg p) \rightarrow q & p, \neg p \nvdash q \\ \nvdash (\Box p \wedge \neg \Box p) \rightarrow q & \Box p, \neg \Box p \nvdash q \ ll \\ \nvdash (\diamond p \wedge \neg \diamond p) \rightarrow q & \diamond p, \neg \diamond p \nvdash q \end{array}$$

These two possible Many-valued Kripke semantics show that the principle of contradiction is independent of the *ex-falso sequitur quod libet* in its two forms, consequential or implicational.

7.5 JKL semantics

Following some ideas of Jaskowski, we can change the definition of truth in a Kripke structure K, by saying that a formula a is true in K iff it is true at *some* world, i.e. there is *some* valuation in which it is true. In case we are working with Many-valued Kripke semantics, this means: there is *some* valuation for which the value of this formula is distinguished.

We will call many-valued with this definition of truth, "JKL-semantics". Such Semantics were introduced in [Béziau 2001].

If we consider the JKL Semantics corresponding to the Semantics MI of the preceding section, with only 1 as distinguished, we have:

$$a, \neg a \nvdash b \qquad \Box a, \neg \Box a \nvdash b \qquad \diamond a, \neg \diamond a \nvdash b$$

but

$$\vdash (a \wedge \neg a) \rightarrow b \qquad \vdash (\Box a \wedge \neg \Box a) \rightarrow b \qquad \vdash (\diamond a \wedge \neg \diamond a) \rightarrow b$$

and

$$\nvdash \neg (a \wedge \neg a) \qquad \nvdash \neg (\Box a \wedge \neg \Box a) \qquad \nvdash \neg (\diamond a \wedge \neg \diamond a).$$

Something that would be interesting is a logic in which the principle of contradiction and the *ex-falso sequitur quod libet* in its two forms are not valid only for modalities. This fits well for example for a logic of beliefs, where someone may have contradictory beliefs without "exploding", but where contradictions explode at the factual level. For this, we need a more sophisticated construction.

7.6 Non truth-functional Kripke semantics

Many-valued semantics are generally truth-functional, that means that they are matrices (see [Béziau 1997] for a detailed account on this question). But it is also possible to introduce non truth-functional many-valued semantics. I have introduced these kind of Semantics in [Béziau 1990] and developed furthermore the subject in [Béziau 2002].

To understand what it means, let us first explain the difference between truth-functional semantics and non truth-functional semantics at the level of bivalent semantics. The set of bivaluations of the semantics of propositional classical logic is the set of homomorphisms from the algebra of formulas and the matrix of truth-functions defined on $\{0, 1\}$. Since the algebra of formulas is an absolutely free algebra, this set can be generated by the set of distributions, i.e. functions assigning 0 or 1 to atomic formulas. A *non truth-functional bivalent semantics* is a semantics where the bivaluations cannot be reduced to homomorphisms between the algebra of formula and an algebra of truth-functions defined on $\{0, 1\}$.

The semantics of classical logic can be presented in two different ways which are equivalent: the usual way with the distributions and the matrix, or by defining directly a set of bivaluations (functions from the whole set of formulas into $\{0, 1\}$ obeying the following conditions:

(1) $\beta\neg(a) = 1$ iff $\beta(a) = 0$
(2) $\beta(a \wedge b) = 1$ iff $\beta(a) = 1$ and $\beta(b) = 1$
(3) $\beta(a \rightarrow b) = 0$ iff $\beta(a) = 1$ and $\beta(b) = 0$

We have a fairly simple example of non truth-functional bivalent semantics, if we replace the condition (1) by the conditions (1'):

(1') if $\beta\neg(a) = 1$ then $\beta(a) = 0$

In this logic, we may have $\beta\neg(a) = \beta(a) = 0$. The logic generated by this condition has been studied in [Béziau 1999a]. Another example of non truth-functional bivalent semantics can be found in [Béziau 1990b]. A general study of logics from the viewpoint of bivalent semantics (truth-functional or non truth-functional) has been developed in [da Costa, *et al.* 1994].

The definition of non truth-functional many-valued semantics is a straightforward generalization: A *non truth-functional many-valued semantics* is a semantics where the valuations cannot be reduced to homomorphisms between the algebra of formula and an algebra of truth-functions defined on a given set of values.

A very simple is the following: we replace Łukasiewicz's condition for negation by the following:

(1') if $\beta\neg(a) = \frac{1}{2}$ then $\beta(a) = \frac{1}{2}$

Now we will construct a non truth-functional many-valued semantics. As in the case of the bivalent semantics for classical propositional logic, truth-

In general this is presented in a rather informal way, where the matrix does not really appear but is described indirectly by means of truth-tables, see [Béziau 2000].

functional bivalent (or many-valued) semantics can be presented in two different way. For example, instead of DEFINITION PWS-W, we can consider an Ipke structure $I =< B, R >$ as a set B of bivaluations assigning to every formula (atomic or molecular) 0 or 1 and a relation R of accessibility between bivaluations.

Then we stipulate that these bivaluations should obey the following conditions:

DEFINITION PWS-W GLOBALIZED
(1) $\beta\neg(a) = 1$ iff $\beta(a) = 0$
(2) $\beta(a \wedge b) = 1$ iff $\beta(a) = 1$ and $\beta(b) = 1$
(3) $\beta(a \vee b) = 1$ iff $\beta(a) = 1$ or $\beta(b) = 1$
(4) $\beta(a \rightarrow b) = 1$ iff $\beta(a) = 0$ or $\beta(b) = 1$
(5) $\beta(\Box a) = 1$ iff for every $\beta' \in B$ such that $\beta R\beta'$, $\beta'(a) = 1$
(6) $\beta(\Box a) = 1$ iff for every $\beta' \in B$ such that $\beta R\beta'$, $\beta'(a) = 1$

Now we replace condition (1) by the following set of conditions:
(1.1.1.) if $\beta(a) = 0$ then $\beta\neg(a) = 1$
(1.2.2.) if $\beta\neg\neg(a) = 1$ then $\beta\neg(a) = 0$
(1.2.3.) if $\beta\neg(a \wedge b) = 1$ then $\beta(a \wedge b) = 0$
(1.2.4.) if $\beta\neg(a \vee b) = 1$ then $\beta(a \vee b) = 0$
(1.2.5.) if $\beta\neg(a \rightarrow b) = 1$ then $\beta(a \rightarrow b) = 0$

This semantics is non truth-functional. In the logic defined by this semantics, we have:

$\nvdash \neg(\Box a \wedge \neg\Box a)$	$\nvdash \neg(\diamond a \wedge \neg \diamond a)$
$\Box a, \neg\Box a \nvdash b$	$\Box a, \neg\Box a \nvdash b$
$\nvdash (\Box a \wedge \neg\Box a) \rightarrow b$	$\nvdash (\diamond a \wedge \neg \diamond a) \rightarrow b.$

but

$\vdash \neg(a \wedge \neg a)$ $\qquad a, \neg a \vdash b$ $\qquad \vdash (a \wedge \neg a) \rightarrow b$

provided there are no modalities in a.

7.7 Conclusion: Many possibilities

We have presented different way to generalize and to combine many-valued and Kripke semantics, and in fact there are still some other possibilities like the semantics developed by Buchsbaum and Pequenos (see e.g. [Buchsbaum *et al.* 2004]) or like the semantics of possible translations developed by Carnielli and Marcos (see e.g. [Carnielli, *et al.* 2002]).

All these tools may be very useful both from an abstract viewpoint of a general theory of logics (see e.g. [Béziau 1994]) and from applications to philosophical problems. For example they can be used, as we have shown, to construct models showing the independency of some properties of negation relatively to some other ones. This is very useful in the field of paraconsistent logic.

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