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A logical analysis of singular terms

A LOGICAL ANALYSIS OF SINGULAR TERMS

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0. Introduction

In philosophy of language, an expression of the type «Brasilia» is considered a *proper name* and an expression of the type «The capital of Brazil» a *definite description*, both being considered *singular terms* by opposition to an expression of the type «Brazilians», considered a *general term.*

A singular term generally *denotes* an object, its *reference* or *denotation*. Some people say that the difference between proper names and definite descriptions is that the latter, besides a reference, have a *meaning*, but not the former.

The meaning can be understood as «the way the reference is given» (Frege 1892, p.26), or «what is grasped when one undestands» (Church 1956, p.7) an expression. It is supposed to be expressed by the structure of the expression and to be objective rather than subjective or psychological. Proper names generally are expressions with no structure or an irrelevant one such as «Rio de Janeiro», which in Portuguese means «River of January».

According to Kripke (cf. Kripke 1980), the difference between definite descriptions and proper names is that the latter are *rigid designators* in the sense that they denote the same thing in all possible worlds, by opposition to definite descriptions whose denotation may vary (we will use hereafter the expression «Kripke's theory» to refer to this view). For example, «The capital of Brazil» in the world of 1950 denotes Rio de Janeiro and in the world of 1999 denotes Brasilia but, according to Kripke's theory, «Rio de Janeiro» denotes the same city in 1950 and in 1999.[1]

These distinctions and related discussions have their origin in the work of Frege and Russell in the logical foundations of mathematics. However nowadays there are nearly no connections between these discussions and mathematical logic. The aim of this paper is to have a look at these central problems of philosophy of language from the standpoint of mathematical logic.

1. Singular terms in formal arithmetic

11. Proper names and definite descriptions in the language *LA* of arithmetic

Let us consider the standard language *LA* of Peano Arithmetic (0, s, +, x) in first-order logic with Russell's description operator ι.

We will consider that:

«0» is a proper name,

«s0» and «$\iota x(\forall y(x \times y = y))$» are (examples of) definite descriptions.

That is to say for us *individual constants* are proper names (in this language «0» is the only proper name), any other *closed term* is a definite description. We take here «constant», «term» and «closed term» in their actual standard technical sense in mathematical logic. We consider thus that the counterpart of natural language singular terms, in a formal language, are closed terms.[2]

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1 We have tried here to define the main terminology in an objective way, independently of any philosophical taste. In the literature, the terminology varies in function of philosophical doctrines, for example some people consider that definite descriptions *are* proper names, that reference is different from denotation, that it is better to call «sense» what we have called meaning, etc.

2 (Haack 1978, p.56) says that «Some formulations of the predicate calculus employ singular terms ('*a*', '*b*' … etc.) as well as variables. (…) Singular terms are usually thought of as the formal analogues of proper names in natural languages».

Let us note that what Haack here calls singular terms are usually called 'constants' in a book of mathematical logic and that the terminology 'singular terms' is rarely used in such a book…

12. Reference in arbitrary *LA*-structures

What are in this framework the denotations of proper names and definite descriptions?

Following Tarski's formal semantics for first-order logic (i.e. model theory) their denotations are relative to an *interpretation* in a given *LA*-structure (i.e. a structure corresponding to the *LA*-language).

In the sandard structure **n**, whose domain is the set of natural numbers $\mathbb{N}$, «0» denotes the standard number zero, that we can name «$\boldsymbol{n}$(0)» to avoid any confusion, «s» denotes the standard successor function, that we can name «$\boldsymbol{n}$(s)», etc. In this structure «s0» and «$\iota x(\forall y(x\times y=y))$» have the same denotation, the number one.

But we can consider an *LA*-structure m, where «0» denotes the number seven, «s» the function x+10 and «+» and «×» the standard addition and multiplication. In this case «s0» and «$\iota x(\forall y(x\times y=y))$» will not denote the same object.

In conclusion: considering the class of all *LA*-structures (taken as the «possible worlds» of philosophy of language[3]), neither «0» nor «s0» nor «$\iota x(\forall y(x\times y=y))$» are rigid designators, in the sense that their references vary.[4]

Therefore if natural language was working in the same way as model theory, Kripke's theory would be meaningless.

It is not obvious, as we assume here as Haack does, that constants of a formal language are counterparts of a proper names of natural language (see the quotation of A.Church below).

[3] «Possible worlds» has a relatively precise meaning in Kripke semantics for modal logic; however in the philosophy of language this expression is used in a rather general and informal way (by Kripke himself), which seems coherent with the present interpretation.

[4] (Church, 1956, p.9) says «We adopt the mathematical usage according to which a proper name of a number is called a *constant*, and in connection with formalized language we extend this usage by removing the restriction to numbers, so that the term *constant* becomes synonymous with *proper name having a denotation*.

However, the term *constant* will often be applied also in the construction of uninterpreted calculi – logistic systems in the sense of #7 – some of the symbols or expressions being distinguished as constants just in order to treat them differently from others in giving the rules of the calculus. Ordinary the symbols or expressions thus distinguished as constants will in fact become proper names (with denotation) in at least one of the possible interpretations of the calculus.»

Thus according to Church, «0» is not a proper name, but only something like «$\boldsymbol{n}$(0)» is. If we adopt strictly this point of view, «$\boldsymbol{n}$(0)» is trivially a rigid designator. To be coherent, Church should consider «$\boldsymbol{n}$(s0)» as a definite description and not «s0». (In fact Church considers definite descriptions proper names; see Church 1956, p.3).

One could say that we must exclude from the notion of possibility of «possible worlds» the possibility of different baptisms (If we admit that in a possible world Saul Kripke could have been named Marlon Brando and vice versa, then Kripke's theory does not work).

But it seems that on the one hand it is not clear how we can do this (in a precise and formal way) and on the other hand that this is not the only trouble, as we will see in the next section.

13. Reference in *LA*-structures which are models of the axioms *AP* of Peano Arithmetic

A framework which will perhaps put model theory closer to natural language would be to consider not arbitrary classes of structures, in our case *LA*-structures, but restricted ones.

So we consider now the case of structures which are models of the standard first-order axioms *AP* of Peano Arithmetic.

Can we say that «0» denotes always the same thing? And what about «s0» and «$\iota x(\forall y\ (x\times y=y))$»?

If a *LA*-structure is a model of *AP*, it obeys certain conditions, «0», «s0» and «$\iota x(\forall y\ (x\times y=y))$» cannot be interpreted in arbitrary ways. For example in all models of *AP*, «s0» and «$\iota x(\forall y(x\times y=y))$» denote one and the same object, because $AP \vdash s0=\iota x(\forall y(x\times y=y))$.

Now let us explain why «0», even in this case of restricted *LA*-structures, does not always denote the same object and therefore is not a rigid designator.

Given any two mathematical structures ***m1*** and ***m2***, how can we say that an object o_1 of the domain of ***m1*** is the same object as an object o_2 of the domain of ***m2***? The «identity» of an object in a mathematical structure is determined by its position in this structure, i.e. the relations it has with other objects of the structure. Therefore o_1 and o_2 are the same object iff there is an isomorphism f from ***m1*** to ***m2*** such that $f(o_1)=o_2$.

It is well-known that first-order arithmetic *AP* is not categorical, that therefore there are two non-isomorphic models of *AP*, for example the standard model n and a non-standard model m. Despite the axioms of arithmetic, «0» does not denote the same object in these two structures, because «0» does not stand in the same position in n and m. For example, in n any object of the domain can be reached from «0» applying the successor function but this is not the case in m.

In fact, as *AP* is incomplete, such differences can be expressed by first-order properties.[5] Consider a formula F which is independent in *AP*, i.e such that: $AP \nvdash F$ and $AP \nvdash \neg F$. Then there exist a model ***m1*** of *AP* in which F is false and a model ***m2*** of *AP* in which $\neg F$ is false, therefore in which F is true. Now given any property P about «0»

[5] In the case of a first-order theory, which is complete but not categorical, the differences cannot be expressed by first-order properties.

such that $AP \vdash P(0)$, in ***m1***, $F \wedge P(0)$ is false and in ***m2***, $F \wedge P(0)$ is true. Notice that $P(0)$ can be a formula that says nothing about «0», for example a tautology and that F can be a formula that says nothing directly about «0» (in the sense for example that «0» does not occur in F).

The same reasoning applied equally to a definite description like «s0». The conclusion is that either «0» and «s0» are rigid designators (case of a categorical theory), or they are not (case of a non categorical theory such as *AP*). Therefore the distinction between proper names and definite descriptions cannot be made in terms of rigid designation and here again Kripke's theory is meaningless.[6]

In natural language, we can have a theory[7] according to which «Brasilia» cannot denote the city of Washington, for example if we have statements in this theory such as «Washington is in the USA», «Brasilia is in Brazil», «USA and Brazil are different countries», etc.

If this theory has just one model, then «Brasilia» and «the capital of Brazil» are both rigid designators.

If this theory admits several models, then «Brasilia» will not denote the same objects in two different possible models, just because these models are different. A Kripkean could argue that this difference does not affect the identity of the city of Brasilia, but affects the reference of «the capital of Brazil». In fact Kripke's theory is based on this mysterious possibility which his opponents consider related to essentialism.

We don't know if essentialism can be a foundation for Kripke's theory. The problem indeed is to find something which can be a foundation for this mysterious possibility.

2. Singular terms in pseudo-formal set theory

21. Formal and pseudo-formal set theory

It is possible to eliminate any singular terms from the language of arithmetic *LA*.[8] A formal philosopher who is convinced that natural language should work as a

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[6] I once heared a famous philosopher of language, explaining Kripke's theory, saying that a typical example of a rigid designator is a mathematical expression like «√2». First notice that «√2» is a definite description rather than a proper name. Second, the reference of «√2» can vary even when it is used by a mathematician who is not a logician and believes that he is working with categorical theories only.

[7] There are various possible counterparts of formal (i.e. axiomatic) theories in natural language; a theory can be a knowledge database, information common to a community of people, etc.

[8] In a language without the description operator terms can be eliminated by simulating individual constants with symbols of monadic predicates, and symbols of functions with symbols of relations. As regards the description operator, Russell himself

formal language could therefore argue that all these philosophical discussions about the distinction between proper name and definite descriptions, etc. are mere sophistry based on the confusions of natural language. His arguments would be similar to those people from Vienna who used to say that most traditional philosophy is the fruit of syntactic confusion. And like these people, he could say that the real work of philosophy is to point out those confusions and to stop the endless discussions arising from them, i.e. in the present time, discussions of philosophy of language, the «Metaphysics» of today.

However if one has a closer look at how mathematicians work including logicians such as set-theorists, he will see that the behaviour of their language is not so different from natural language and that perhaps it is the present logical formalization of mathematics, rather than natural language, which has to be transformed.

What happens in everyday mathematics is exactly the contrary of elimination of singular terms. Singular terms are introduced and they work in a way not so much different as they work in natural language.

There is a big gap between the formal and informal treatement of mathematical theories; this is clear for example in the case of number theory and its formal counterpart, Peano arithmetic.

An interesting case is the one of set theory where the gap is not so big. On the one hand we have formal set theory, on the other hand something that we can call *pseudo-formal* set theory.

In all books of set theory, the famous Weil's symbol «∅» for the empty set is introduced. However in most books dealing with set theory as a formal theory and not only as a «naive» theory, this symbol is not properly introduced as a part of formal language. This is typical of pseudo-formal set theory. To turn this pseudo-formal theory completely formal, one has to consider set-theory with the operator of description and to define «∅» *in* the language with a Leśniewski's style definition, for example:

$$\emptyset =_{\mathrm{Def}} \iota x\ (\forall y(y \in x \rightarrow y \neq y))$$

In principle the formal language of set theory is extremely poor. There is only one symbol of binary relation «∈» and no singular terms. However a lot of singular terms besides «∅» are introduced informally.[9] Let us see how they works.

showed how it can be eliminated.

[9] In principle it seems that there are no difficulties to turn pseudo-formal set theory into formal set theory. But a lot of things have to be checked. The operator of description is a *vbto* (variable binding term-forming operator) and one has to check that the usual syntax and semantics of first-order logic and the correlative results can be extended to *vbtos*. Such work has been carried out by Corcoran, Herring, Hatcher and da Costa. References can be found in (da Costa/Mortensen, 1983) which is a kind of survey.

22. «$\aleph_0$» and «ω»: two proper names with different meanings but the same reference

Let us analyse the case of $\aleph_0$ and ω. They are in some sense two different proper names with the same denotation. Why to use two different names to refer to the same object? This seems against the idea that mathematics is a perfect science wishing to avoid ambiguity. In fact in mathematics, as in natural language, this practice is very common (as well as using the same name for different things).

We can say that «$\aleph_0$» and «ω» are two different proper names that have the same reference but different meanings. We can say that their meanings are given by two co-designative definite descriptions, the first refering to the object as a cardinal and the second as an ordinal. They are two different ways to look at the same object.

If one should like to «fill the gap» between pseudo-formal set theory and formal set theory, he should introduce «ω» by definition:

$$\omega =_{\text{Def}} \iota x(Fx)$$

where F is a formula saying that ω is the first infinite ordinal. There are several equivalent ways to define ω. In fact any formula equivalent to F modulo *ZFC* does the job.

So what is the meaning of the proper name «ω»? If we consider that it is the class of all equivalent definite descriptions, then «$\aleph_0$» and «ω» have the same meaning, and this does not fit with the idea of the mathematician who introduces two different names. If we consider that this is only a description, this seems too restrictive, because there are several ways to conceive ω as an ordinal (there are for example several equivalent definitions of the notion of ordinal).

Therefore the meaning of the proper name «ω» seems to be a certain intermediate class of co-designative logically equivalent definite descriptions. This view is close to Wittgenstein's analysis of the proper name «Moses» (cf. Wittgenstein, 1953, #79). The main difference concerns «logically equivalent». In natural language there is instead a notion of equivalence which is much fuzzier.

It is is difficult to define rigorously this class for at least two reasons:

(a) If $\iota x(Fx)$ is part of the meaning of «ω», should we consider that so is $\iota x(Fx \wedge Fx)$?

(b) The meaning is something which changes according to the advances of mathematics, in particular proofs of new theorems. If one proves that F is equivalent in *ZFC* to an apparently very different formula G, then $\iota x(Gx)$ would become part of the meaning of «ω».

23. What are the references of «$\aleph_0$», «ω» and «$\aleph_1$»?

We have said that «$\aleph_0$» and «ω» have the same reference. What does it mean exactly? This means that in every given model of *ZFC* they denote the same object, and not that in two different models of *ZFC* they denote the same object. *ZFC* is not categorical and «ω» and «$\aleph_0$», as happens with «0» in the case of Peano Arithmetic, are not rigid designators in the sense that we have explained in section 13 above.

Let us consider the sentence:

DuPont wants to know if $\aleph_0 = \omega$.

Inspired by Frege, we can say that what DuPont wants to know is if «$\aleph_0$» and «ω», which have different meanings, have the same denotation in any given model of *ZFC*. We can consider therefore that the true denotation of a proper name like «$\aleph_0$» is the set of all its denotations in all models of *ZFC*. We will call such true denotation, its Denotation. Such a Denotation can be considered the set of all equivalent definite descriptions which define a given proper name. (Therefore this view does not necessarily commit one to Platonism or an ontology of abstract objects.)

Now if two proper names have not the same Denotation, there are two possibilities:

(a) they have different denotations in every given model of *ZFC*, as is the case for example as regards «$\aleph_0$» and «$\aleph_1$».

(b) they have different denotations in some models of *ZFC* and the same denotation in some models of *ZFC*, case of «$\aleph_1$» and «$2^{\aleph 0}$».

Accordingly the sentence:

DuPont wants to know if $\aleph_1 \neq 2^{\aleph 0}$.

can be interpreted in two different ways.

24. «$\aleph_1$» and «$2^{\aleph 0}$»: the interplay between meaning and reference

According to this view the Denotation of a proper name like «$\aleph_1$» is something difficult to define or to catch, something one could say *inaccessible*. What the set-theorist is trying to do is to precise the meaning of «$\aleph_1$», trying to compare it with other proper names like «$2^{\aleph 0}$». As we have said, for us, the meaning of a proper name is a set of definite descriptions. The meaning of «$2^{\aleph 0}$» is relatively clear because among the set of definite descriptions corresponding to «$2^{\aleph 0}$» there are several well intelligible entities, such as the cardinality of the set of reals, etc. But we know few things about «$\aleph_1$», we know that it is the next infinite cardinal, but we don't know which kind of well-known sets have this cardinality.

To precise absolutely the meaning of a proper name like «$\aleph_1$» would be to get its Denotation, which is something impossible. But one can get more information about it by identifying or differentiating it from another proper name (i.e. cluster of definite descriptions) like «$2^{\aleph 0}$». The set-theorist has an idea about «$\aleph_1$» and his idea is not fixed, the meaning of «$\aleph_1$» is changing, mainly by proofs of new results.

The reference of «$\aleph_1$» considered its Denotation seems to be fixed and therefore one could claim that «$\aleph_1$» is trivially a rigid designator. But in fact the reference of «$\aleph_1$» can also changed, if we modify the axioms of set theory. Some people are looking for axioms from which it will be possible to prove the continuum hypothesis, i.e. according to which «$\aleph_1$» would have the same Denotation as «$2^{\aleph 0}$».

3. Conclusion

It seems to us that proper names in natural language work in a similar way as proper names in pseudo-formal set theory:

- they are abbreviations of a cluster of a fuzzy changing set of co-designative definite descriptions;

- their meaning is the set (of meanings) of these definite descriptions and therefore is not stable (When the meaning of a proper name changes radically, the name may change accordingly and we have a new «baptism», both in mathematics and in natural language);

- their reference is the set of all equivalent definite descriptions, and may vary in function of the notion of equivalence, in function of the underlying theory, therefore proper names are not rigid designators.

A definite description is a particular case of a proper name, i.e. when the cluster is a singleton. The meaning of a definite expression is expressed by its structure and its Denotation is the set of possible interpretations of this structure. For example we can say that the Denotation of «The capital of Brazil» are the cities of Rio de Janeiro, Brasilia and Salvador at the times when they respectively were capitals of Brazil.

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