Bivalent Semantics for De Morgan Logic (The Uselessness of Four-valuedness)

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Dedicated to Newton da Costa for his 79th birthday

ABSTRACT. In this paper we present a bivalent semantics for De Morgan logic in the spirit of da Costa's theory of valuation showing therefore the uselessness of four-valuedness - the four-valued Dunn-Belnap semantics being ordinarily used to characterize De Morgan logic. We also present De Morgan logic in the perspective of universal logic, showing how some general results connecting bivaluations to sequent rules and reducing many-valued matrices to non-truth functional bivalent semantics work.

1 De Morgan logic in the perspective of universal logic

In this paper we present a systematic study of a very simple and nice logical structure, De Morgan logic. This is a logic with a negation which is both paraconsistent and paracomplete, that is to say neither the principle of contradiction, nor the principle of excluded middle are valid for De Morgan negation, but all De Morgan laws hold, the reason for the name. This logic shows therefore the independence of the principle of contradiction and the principle of excluded middle relatively to De Morgan laws [1].

This logic is not new. It is connected with De Morgan lattices which can be traced back to Moisil [22] and which have been called quasi-boolean algebras by Rasiowa, [13], distributive i-lattices by Kalman [20], and have been especially studied by the school of Antonio Monteiro in Bahia-Blanca, Argentina [23].

[1] Negations which are both paraconsistent and paracomplete were called by da Costa *non-alethic*, we have proposed to used instead the adjective *paranormal* in order to keep the paraterminology. De Morgan negation is a good example of paranormal negation, it can reasonably be considered as a negation, due to the fact that it obeys De Morgan laws.

If we factor a De Morgan logic by the relation of logical equivalence, we get a De Morgan lattice, and an intuitive semantics for De Morgan logic is a matrix corresponding to a very De Morgan lattice, a four-valued lattice. This kind of semantics is connected with Dunn-Belnap four-valued semantics [3]. Michael Dunn especially made the connections between De Morgan lattices and logical systems [17]. This connection was later on studied by J.M.Font and V.Verdu [19] [18], O.Arielli and A.Avron [2] and A.P.Pynko [26].

We are facing here a phenomenon similar to the one happening with classical propositional logic: at the same time the factor structure of classical propositional logic is a Boolean algebra and its matrix semantics is the smallest Boolean algebra.

We show in this paper that it is also possible to construct a bivalent semantics for De Morgan logic, along the line of Newton da Costa's theory of valuation [16]: it is a non-truth functional bivalent semantics. And we also show how to establish the connection between this bivalent semantics and the four-valued De Morgan matrix.

In some previous papers we have shown how we can establish a close connection between non-truth functional bivalent semantics and sequent rules [12]. We also have proven some general results of reduction of semantics to bivalent semantics [7]. This is part of a general study of logical structures, we have called *universal logic* [5].

This paper is the opportunity to illustrate these general results of universal logic, as we already did for Łukasiewicz logic $L3$ [9] and also to point out some problems that could be usefully clarified by universal logic.

2 De Morgan logical structure

DEFINITION 1. A *De Morgan logic* is a structure $\mathcal{M} = \langle \mathcal{F}; \vdash \rangle$, where

- $\mathcal{F}$ is an is an absolutely free algebra $\langle \mathbb{F}; \wedge, \vee, \neg \rangle$ whose domain $\mathbb{F}$ is generated by the functions $\wedge, \vee, \neg$ from a set of atomic formulas $\mathbb{A} \subset \mathbb{F}$.
- $\vdash$ is a structural consequence relation obeying, besides the usual axioms for classical conjunction and disjunction, the following axioms:

[¬∧] $\neg(a \wedge b) \dashv\vdash \neg a \vee \neg b$
[¬∨] $\neg(a \vee b) \dashv\vdash \neg a \wedge \neg b$
[¬¬] $a \dashv\vdash \neg\neg a$
where $x \dashv\vdash y$ means $x \vdash y$ and $y \vdash x$.

Defining a logic in this way is typical of the Polish approach [24], although in Poland people generally prefer to use the notion of consequence operator rather than the notion of consequence relation, but this is trivially equivalent. The notion of structural consequence operator is due to Łoś and Suszko [21] and it is the continuation of Tarski's theory of consequence operator which began at the end of the 1920s [27].

One has to be aware that such kind of definition is not proof-theoretical, it is not a system of deduction. The use of the word *axiom* and of the symbol $\vdash$ may lead to such kind of confusion. But as it is known the word *axiom* is also used in model theory. Such a definition has to be seen as a definition of for example the model-theoretical definition of structures of order. What we call a De Morgan logic is a model of the above group of axioms which defines the meaning of the relation $\vdash$ and of the functions $\wedge, \vee, \neg$.

The kind of definition we are using here is of the same type for example as the definition of a De Morgan lattice. Moreover there is a strong connection between these two definitions, since the factorization of a De Morgan logic by the relation $\dashv\vdash$, which is a congruence, is a De Morgan lattice.

Let us recall the definition of a De Morgan lattice.

DEFINITION 2. A *De Morgan lattice* is a distributive lattice $\langle \mathbb{E}; \cap, \cup, \sim \rangle$ where the unary operator obeys the two following axioms:

$\sim (a \cap b) = \sim a \cup \sim b$

$a = \sim \sim a$

3 Sequent System for De Morgan logic

DEFINITION 3. A *De Morgan sequent system* $\mathcal{LM}$ is a sequent system which has the same rules for conjunction and disjunction as the sequent system for classical propositional logic and which has the following rules for negation [2]:

$$\frac{\neg a \Rightarrow \qquad \neg b \Rightarrow}{\neg(a \wedge b) \Rightarrow}\ [\neg_l \wedge] \qquad\qquad \frac{\Rightarrow \neg a, \neg b}{\Rightarrow \neg(a \wedge b)}\ [\neg_r \wedge]$$

$$\frac{\neg a, \neg b \Rightarrow}{\neg(a \vee b) \Rightarrow}\ [\neg_l \vee] \qquad\qquad \frac{\Rightarrow \neg a \qquad \Rightarrow \neg b}{\Rightarrow \neg(a \vee b)}\ [\neg_r \vee]$$

$$\frac{a \Rightarrow}{\neg\neg a \Rightarrow}\ [\neg_l \neg] \qquad\qquad \frac{\Rightarrow a}{\Rightarrow \neg\neg a}\ [\neg_r \neg]$$

[2] We don't know who was the first to present such sequent rules, but they can be found in particular in [19] [1] [26].

To make things clearer we have written the rules without the contexts, but $\mathcal{LM}$ is contextually standard. It has also the same structural rules as the system for classical propositional logic, including the cut rule. Cut-elimination for $\mathcal{LM}$ can easily be proven, following a method similar to the one presented in [4].

The rules of $\mathcal{LM}$ don't have the subformula property, but they have something analogous: the subnegformula property. A subnegformula of a formula is a proper subformula or a negation of a proper subformula.

From cut-elimination and the subnegformula property it results the decidability of the logical structure generated in the usual way by $\mathcal{LM}$.

THEOREM 4. *A logical structure generated by a $\mathcal{LM}$ system of sequents is a De Morgan logic.*

Proof. We know that a logical structure generated by a structurally standard sequent system is a structural consequence relation [3]. We have to show furthermore that all axioms of Definiton 1 are valid in a logic generated by a $\mathcal{LM}$ system of sequents. We will study just the case of axiom $[\neg\wedge]$. It is enough to prove that the sequents $\neg(a\wedge b) \Rightarrow \neg a\vee\neg b$ and $\neg a\vee\neg b \Rightarrow \neg(a\wedge b)$ are derivable in a $\mathcal{LM}$ system of sequents.

$$\dfrac{\dfrac{\neg a \Rightarrow \neg a, \neg b}{\neg a \Rightarrow \neg a \vee \neg b} \qquad \dfrac{\neg b \Rightarrow \neg a, \neg b}{\neg b \Rightarrow \neg a \vee \neg b}}{\neg(a \wedge b) \Rightarrow \neg a \vee \neg b}\ [\neg_{l\wedge}]$$

$$\dfrac{\dfrac{\neg a \Rightarrow \neg a \qquad \neg b \Rightarrow \neg b}{\neg a \vee \neg b \Rightarrow \neg a, \neg b}}{\neg a \vee \neg b \Rightarrow \neg(a \wedge b)}\ [\neg_{r\wedge}]$$

■

What is more difficult is to prove the converse of this theorem:

THEOREM 5. *A De Morgan logic can be generated by a $\mathcal{LM}$ system of sequents.*

Proof. To prove this theorem, there are two parts.

[3] *Structural* here is used in two different ways. A structural consequence relation is a relation invariant by substitutions. Using schema of rules we necessarily generate such a structural consequence relation. A structurally standard sequent system is a system having all the structural rules and standard contextual behaviour – see [12].

- The first part consists in a completeness theorem between the Definition 1 of De Morgan logic and a sequent systems close to it, $\mathcal{LM}_0$, which is a system of sequents with the standard structural rules, and with rules which are the most direct translations of the axioms for negation into sequent rules, for example the (model theoretical) axiom $[\neg\wedge]$ of Definition 1 is translated into the two following (proof-theoretical) axioms of the sequent system $\mathcal{LM}_0$: $\neg(a \wedge b) \Rightarrow \neg a \vee \neg b$ and $\neg a \vee \neg b \Rightarrow \neg(a \wedge b)$.

- The second part is just to prove that the rules of $\mathcal{LM}$ are derivable rules of $\mathcal{LM}_0$.

In this paper we will not deal with the first part of this proof. It is possible to prove a general completeness theorem connecting logical structures to systems of sequents. This is universal logic. Details of such connection will be given in another paper.

The second part of the proof is quite easy. We present the part showing how we can derive in $\mathcal{LM}$ the rules $[\neg_{l\wedge}]$ and $[\neg_{r\wedge}]$ from the axiom $[\neg\wedge]$.

$$\dfrac{\neg(a \wedge b) \Rightarrow \neg a \vee \neg b \text{ [Axiom]} \qquad \dfrac{\neg a \Rightarrow \qquad \neg b \Rightarrow}{\neg a \vee \neg b \Rightarrow}}{\neg(a \wedge b) \Rightarrow}\,[cut]$$

$$\dfrac{\dfrac{\Rightarrow \neg a, \neg b}{\Rightarrow \neg a \vee \neg b} \qquad \neg a \vee \neg b \Rightarrow \neg(a \wedge b) \text{ [Axiom]}}{\Rightarrow \neg(a \wedge b)}\,[cut]$$

■

4 Bivalent semantics for De Morgan logic

4.1 A non-truth functional bivalent semantics

DEFINITION 6. We consider the set of functions $\mathbb{B}$ from $\mathbb{F}$ to the set $\{0, 1\}$ defined by the usual conditions for conjunction and disjunction and the following conditions for negation:

$[\![\neg\wedge]\!]$ $\beta(\neg(a \wedge b)) = 1$ iff $\beta(\neg a) = 1$ or $\beta(\neg b) = 1$
$[\![\neg\vee]\!]$ $\beta(\neg(a \vee b)) = 1$ iff $\beta(\neg a) = 1$ and $\beta(\neg b) = 1$
$[\![\neg\neg]\!]$ $\beta(\neg\neg a) = 1$ iff $\beta(a) = 1$

Note that this is typically a generalized bivalent semantics in the sense of da Costa [16] : this set of bivaluations is not generated by distributions of truth-values for atomic formulas, and it is not a set of homomorphisms between the set of formulas and some algebra of similar type of a logical matrix.

DEFINITION 7. The semantical consequence relation $\models_2$ is defined in the usual way:

$T \models_2 k$ iff for every $\beta \in \mathbb{B}$, if $\beta(j) = 1$ for every $j \in T$, then $\beta(k) = 1$.

4.2 Truth-Tables

Following the idea of da Costa and Alves [15], we can build some truth-tables based on this semantics.

DEFINITION 8. A truth-table is a table with a finite number of columns and lines such that that on the first line in each column we have a formula and on the other lines (proper lines) in each column we have 0 or 1 obeying the following conditions:

- each proper line of the table can be extended into a bivaluation β from $\mathbb{F}$ to $\{0, 1\}$
- for any bivaluation β, there is a proper line of the table such that $\beta(x) = y$, for any x, x being a formula given by the first line, and y being 0 or 1 according to the given line.

EXAMPLE 9. The following truth-table shows that for the atomic formulas p and q we have $\neg(p \wedge q) \models_2 \neg p \vee \neg q$ and $\neg p \vee \neg q \models_2 \neg(p \wedge q)$:

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
0	0	0	0	0	0	0
0	0	0	1	0	1	1
0	0	1	0	0	1	1
0	0	1	1	0	1	1
0	1	0	0	0	0	0
0	1	0	1	0	1	1
0	1	1	0	0	1	1
0	1	1	1	0	1	1
1	0	0	0	0	0	0
1	0	0	1	0	1	1
1	0	1	0	0	1	1
1	0	1	1	0	1	1
1	1	0	0	1	0	0
1	1	0	1	1	1	1
1	1	1	0	1	1	1
1	1	1	1	1	1	1

This table has some additional features, corresponding to the following definition:

DEFINITION 10. A truth-table is said to be full if the set of formulas of the first line is closed by the subnegformula property, i.e. it contains all proper subformulas and negations of proper subformulas of this set of formulas.

THEOREM 11. *For any formula it is possible to construct a full truth-table having on the first line the formula and the set of its subnegformulas.*

Proof. We have first to give a method to build a table and then to show that this table is a full truth-table. This can been done along the same line as for the construction of truth-tables for the paraconsistent logic C1, see e.g. [4]. ■

5 Adequacy of the bivalent semantics

THEOREM 12 (Soundness). *If* $T \vdash k$ *then* $T \models_2 k$

Proof. Straightforward, we leave it to the reader. ■

THEOREM 13 (Completeness). *If* $T \nvdash k$ *then* $T \not\models_2 k$

Proof. If $T \nvdash k$ then there is, according to Lindenbaum-Asser theorem [25] [11] a relatively maximal extension of T in k, i.e. a set of formulas V such that

- $T \subseteq V$
- $V \nvdash k$
- $W \vdash k$ for any strict extension W of V.

We just have to show that the characteristic function β_V of V is a bivaluation, then we will have $T \not\models_2 k$.
We show that the characteristic function β_V of V obeys the condition
$[\![\neg\wedge]\!]$ $\beta(\neg(a \wedge b)) = 1$ iff $\beta(\neg a) = 1$ or $\beta(\neg b) = 1$
leaving the other cases for the reader.

If $\beta_V(\neg a) = 1$ or $\beta_V(\neg b) = 1$, then $V \vdash \neg a$ or $V \vdash \neg b$, then by the rule $[\neg_{r\wedge}]$, $V \vdash \neg(a \wedge b)$ then $\beta_V \neg(a \wedge b) = 1$

If $\beta_V(\neg a) = 0$ and $\beta_V(\neg b) = 0$, then $V \nvdash \neg a$ and $V \nvdash \neg b$, then, since V is maximal in k, $V, \neg a \vdash k$ and $V, \neg b \vdash k$, then by the rule $[\neg_{l\wedge}]$, $V, \neg(a \wedge b) \vdash k$, therefore $V \nvdash \neg(a \wedge b)$, therefore $\beta_V \neg(a \wedge b) = 0$. ■

6 Four-valued semantics for de Morgan logic

6.1 The four-valued De Morgan matrix

DEFINITION 14. We consider a set of four values $\{0^-, 0^+, 1^-, 1^+\}$ partially ordered as following
$0^- \prec 0^+ \prec 1^+$
$0^- \prec 1^- \prec 1^+$
Using this partially order we define the following three functions from $\{0^-, 0^+, 1^-, 1^+\}$ to $\{0^-, 0^+, 1^-, 1^+\}$.

$\vec{\wedge}(x, y) = inf(x, y)$

$\vec{\vee}(x, y) = sup(x, y)$

$\vec{\neg}(x) = x$ if $x \in \{0^+, 1^-\}$
$\vec{\neg}(0^-) = 1^+$
$\vec{\neg}(1^+) = 0^-$

The structure $\langle\{0^-, 0^+, 1^-, 1^+\}; \vec{\wedge}, \vec{\vee}, \vec{\neg}\rangle$ is a finite De Morgan lattice, we will call it the four-valued De Morgan lattice.

The functions $\vec{\wedge}, \vec{\vee}, \vec{\neg}$ can be called truth-functions and be defined more visually by the following truth-tables.

$\vec{\wedge}$	0^-	0^+	1^-	1^+
0^-	0^-	0^-	0^-	0^-
0^+	0^-	0^+	0^-	0^+
1^-	0^-	0^-	1^-	1^-
1^+	0^-	0^+	1^-	1^+

TRUTH-TABLE FOR CONJUNCTION

$\vec{\vee}$	0^-	0^+	1^-	1^+
0^-	0^-	0^+	1^-	1^+
0^+	0^+	0^+	1^+	1^+
1^-	1^-	1^+	1^-	1^+
1^+	1^+	1^+	1^+	1^+

TRUTH-TABLE FOR DISJUNCTION

	$\vec{\neg}$
0^-	1^+
0^+	0^+
1^-	1^-
1^+	0^-

TRUTH-TABLE FOR NEGATION

DEFINITION 15. We consider the set of homomorphisms between the absolutely free algebra of formulas and the four-valued De Morgan lattice. This defines a set of quadrivaluations $\mathbb{T}$ from $\mathbb{F}$ to $\{0^-, 0^+, 1^-, 1^+\}$.

DEFINITION 16. We consider as a logical matrix the four-valued de Morgan lattice together with $\{1^-, 1^+\}$ as set of designated values.

DEFINITION 17. The semantical consequence relation $\models_4$ is defined in the usual way:

$T \models_4 k$ iff for every $\theta \in \mathbb{T}$, if $\theta(j)$ is designated for every $j \in T$, then $\theta(k)$ is designated.

This four-valued semantics is sound and complete for De Morgan logic. There are several ways to prove that. We will use here a translation between the four-valued semantics into the two-valued semantics.

What is interesting with this four-valued semantics is that it is based on an algebra which is similar to the algebra which is the factor structure of a De Morgan logical structure we get using the relation of logical equivalence. For the semantics however what is used is a finite De Morgan lattice, the four-valued one.

6.2 Translation of the four-valued semantics into the two-valued semantics

A way to reduce any semantics to a non-truth-functional bivalent semantics has been studied in particular in [7]. The example of reduction presented here is a straightforward applications of this method.

DEFINITION 18. Given a function β of the bivalent semantics, we define a quadrivaluation θ_β in the following way
$\theta_\beta(k) = 0^-$ iff $\beta(k) = 0$ and $\beta(\neg k) = 1$
$\theta_\beta(k) = 0^+$ iff $\beta(k) = 0$ and $\beta(\neg k) = 0$
$\theta_\beta(k) = 1^-$ iff $\beta(k) = 1$ and $\beta(\neg k) = 1$
$\theta_\beta(k) = 1^+$ iff $\beta(k) = 1$ and $\beta(\neg k) = 0$

THEOREM 19. *This is a bijection between* $\mathbb{B}$ *and* $\mathbb{T}$ *such that:*
$\theta_\beta(k)$ *is designated iff* $\beta(k) = 1$.

Proof. Straightforward on the complexity of formulas. ■

As a corollary we have the following result:

THEOREM 20. $T \models_2 k$ *iff* $T \models_4 k$.

The fact that the four-valued matrix semantics for De Morgan logic can be reduced to a bivalent semantics does not mean that the four-valued matrix semantics has no interest - for a discussion about this, see [14].

The subtitle of our paper is a kind of joke referring to the paper by O. Arielli and A. Avron, called “The value of four values” [2]. Nevertheless it is true that in some sense we don’t need four-values to deal with De Morgan logic.

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