RULES, DERIVED RULES, PERMISSIBLE RULES

AND

THE VARIOUS TYPES OF SYSTEMS OF DEDUCTION

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Abstract. We first define the notions of rule, derived rule, permissible rule for any system whatsoever ; then we state the exact distinction between two kinds of systems strongly connected: first level and second level systems. This abstract setting is illustrated by the example of the implicative intuitionistic propositional logic. Then we have a look at some historical developments.

Introduction

As logic is expanding most confusions arise from a lack of a general framework. Fundamental concepts which at first had a precise meaning have progressively been completely distorted and are used in various ways which are not coherent simultaneously. This problem is especially vivid for concepts like those of rule and system of deduction which are basic tools for philosophical logic as well as for mathematical logic. We provide here a general abstract setting and we give clear analyses of some well-known problems such as the status of the cut rule in a cut-free system, the classification of systems, the way to present the modus ponens.

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ABSTRACT SETTING

1. Rules and derived rules

Let $\mathfrak{X}$ be a set, a *rule* R over $\mathfrak{X}$ is a pair ⟨X;x⟩ where X is a subset of $\mathfrak{X}$ and x is an element of $\mathfrak{X}$. The elements of X are called *premises* of R and x *conclusion* of R. A rule whose set of premises is empty is called an *axiomatic rule*. The following symbolisation will be used for representation of rules:

$$\frac{x1 \; \ldots \; x\alpha \; \ldots}{x},$$

up the premises, below the conclusion ; note that we are using the notion of order in this symbolization only for naming.

A *proof* of x *from* the set of *assumptions* X over a set of rules $\mathfrak{R}$ is a sequence ⟨x1;...;xα;....;x⟩ such that each of its member is the conclusion of a rule of $\mathfrak{R}$ whose premises precede it or is an assumption.

A *derived rule* of $\mathfrak{R}$ is a rule ⟨X;x⟩ such that there exists a proof of x from the set of assumptions X.

Given a set $\mathfrak{R}$ of rules, it is clear that every rule of $\mathfrak{R}$ is a derived rule of $\mathfrak{R}$, such kind of derived rules will be called primitive rules.

Now let us write $X \vdash x$ to say that there is a proof of x from the set of assumptions X. A *system of deduction* $\mathfrak{S}$ is a triple $\mathfrak{S}=\langle\mathfrak{X};\mathfrak{R};\vdash\rangle$. For the sake of simplicity we will speak of the system $\mathfrak{R}$ over $\mathfrak{X}$, or of the system $\mathfrak{R}$, or of $\mathfrak{R}$.

We call the relation $\vdash$ the *logic* of (the system) $\mathfrak{R}$, and the sublogic of the logic of $\mathfrak{R}$ which contains only the pairs ⟨X;x⟩ such that X=∅, the *semi-logic* of $\mathfrak{R}$; the semi-system $\mathfrak{R}$ is the system $\mathfrak{R}$ considered with its semi-logic instead of its logic.

It is clear that $\langle X;x\rangle$ is a derived rule of $\mathfrak{R}$ iff $X\vdash x$: the logic of $\mathfrak{R}$ is the set of derived rules of $\mathfrak{R}$ and the semi-logic of $\mathfrak{R}$ is the set of axiomatic derived rules of $\mathfrak{R}$.

2. Logics and permissible rules

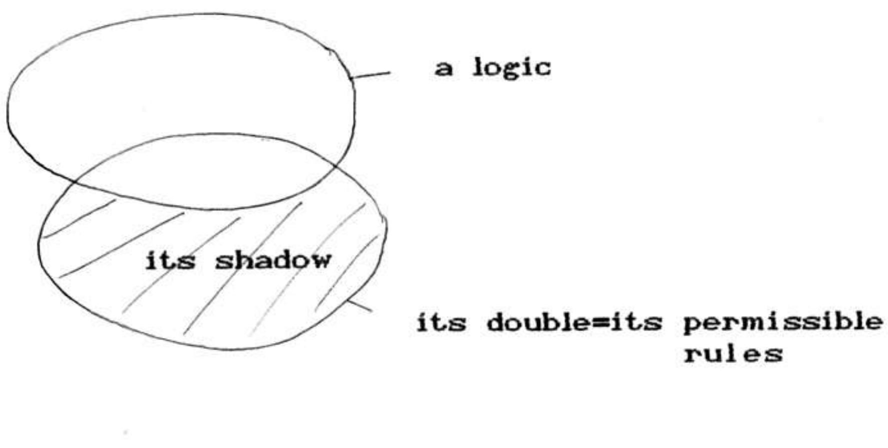
A *logic* $\mathfrak{L}$ over $\mathfrak{X}$ is a pair $\langle\mathfrak{X};\vdash\rangle$ where $\vdash$ is a set of rules over $\mathfrak{X}$. $X\vdash x$ is written for $\langle X;x\rangle\in\vdash$.

Given a logic over $\mathfrak{X}$, *a permissible rule* $\langle\{x1,\dots,x\alpha,\dots\};x\rangle$ is a rule such that:

if $\emptyset\vdash x1$ and ... and $\emptyset\vdash x\alpha$ and ... then $\emptyset\vdash x$

Example: given a logic over $\mathfrak{X}$ and x an element of $\mathfrak{X}$ $\langle\{x\};x\rangle$ is a permissible rule.

The set of permissible rules of a logic over $\mathfrak{X}$ is itself a logic over $\mathfrak{X}$, we will call it its *double*. The logic over $\mathfrak{X}$ which contains every pair $\langle X;x\rangle$ which is in the double of $\mathfrak{L}$ but not in $\mathfrak{L}$ will be called the *shadow* of $\mathfrak{L}$. If the shadow of a logic is non-empty we will say that it is a *shadowy logic*, if not it is a *shadeless logic* ; or: a logic has a shadow, has no shadow.



A semi-logic over a non-empty set $\mathfrak{X}$ is always a shadowy logic because if x is in $\mathfrak{X}$ $\langle\{x\};x\rangle$ is a permissible rule but $\{x\}\nvdash x$.

Given a logic, an expression about $\vdash$ will be called a *law*. Example of law:

folding law: for every x,x1,...,xα,...

$$\text{if} \quad \emptyset \vdash x1 \text{ and } \ldots \text{ and } \emptyset \vdash x\alpha \text{ and } \ldots \quad \text{then} \quad \emptyset \vdash x$$

$$\text{iff}$$

$$\{x1,\ldots,x\alpha,\ldots\} \vdash x.$$

We will call the *unfolding law*, the first half of this law (=if) and the *enfolding law*, the second half (=only if). And we will speak about *folding logic*, etc.

Using this terminology we have: a logic is an unfolding logic iff it is a sublogic of its double, an enfolding logic iff it has no shadow and a folding logic iff it is the same as its double.

Another example of law is:

cut law: for every x, y, X, Y,

$$\text{if} \quad X \vdash x \text{ for every } x \in Y \text{ and } Y \vdash y \quad \text{then} \quad X \vdash y.$$

*1 *Every cut logic is an unfolding logic.*

A *logic* is said to be *determined* by a system iff it is the logic of this system.

One more law:

identity law: for every x and X such that $x \in X$, $X \vdash x$.

A logic for which the identity and the cut laws hold will be called a *normal logic*.

*2 ▸ *A logic over* $\mathfrak{X}$ *is normal iff it can be determined by a system over* $\mathfrak{X}$.

3. Systems and permissibility

The logic of a system is a cut logic (*2) thus it is an unfolding logic (*1), every derivable rule is permissible.

Given a system $\mathfrak{R}$ we can speak about the permissible rules of $\mathfrak{R}$, that means the permissible rules of its logic, but we must keep in mind that this set of permissible rules is independent of $\mathfrak{R}$ in the following sense: if we have a logic which has a shadow, this shadow will always be there: in any system which determines this shadowy logic there are permissible non-derivable rules.

*3 ▶ *If two enfolding systems have the same semi-logic (=axiomatic derived rules) they have the same logic (=derived rules).*

*4 ▶ *A rule is permissible for a system iff in adding it to the system the new system so-obtained has the same semi-logic.*

A *system* $\mathfrak{R}$ is said to be *derivable* from a system $\mathfrak{R}'$ iff every primitive rule of $\mathfrak{R}$ is a derived rule of $\mathfrak{R}'$. $\mathfrak{R}$ and $\mathfrak{R}'$ are *interderivable* iff $\mathfrak{R}$ is derivable from $\mathfrak{R}'$ and vice versa.

*5 ▶ *Two systems are interderivable iff they have the same logic (=derived rules).*

But two systems can have the same semi-logic and be non-interderivable: if a primitive rule of the first two system is a permissible non-derivable rule of the second.

4. Rising : second level systems and metalogic

Given a set $\mathfrak{X}1$, the set of all rules of $\mathfrak{X}1$ is called $\mathfrak{X}2$.

A *first level logic* (using symbolism: a 1-logic) is a logic over $\mathfrak{X}1$.

A *second level logic* (= a 2-logic) is a logic over $\mathfrak{X}2$.

Similarly we define the concepts of first and second level rule and system. A second level rule will be represented like this:

$$\frac{X1 \succ x1 \ \ldots \ X\alpha \succ x\alpha \ \ldots}{X \succ x},$$

where $X \succ x$ is a symbolization of $\langle X;x \rangle$.

Given a logic, laws about $\mathfrak{L}$ (ie, expressions about $\vdash$) and true laws for $\mathfrak{L}$ form a kind of general *metalogic*, the problem to know whether it can be considered as a logic or not will not be treated here. We will limit ourselves to a restricted part of metalogic.

Among all the laws about $\vdash$ we will consider only those which have the following form:

if $X1 \vdash x1$ and ... and $X\alpha \vdash x\alpha$ and ... then $X \vdash x$.

If such a law holds for a logic $\mathfrak{L}$ it will be called a *2-metarule* of $\mathfrak{L}$.

Note that the cut, identity, enfolding laws are not exactly of this form but they can be considered as sets of laws of this form. The unfolding law cannot be considered in this way.

Given a 1-logic, this kind of metalogic can be treated in terms of 2-logic.

The *2-metalogic* $2\mathfrak{ML}=\langle \mathfrak{X}2;\vdash_2 \rangle$ of a 1-logic $\mathfrak{L}=\langle \mathfrak{X}1;\vdash_1 \rangle$ is the following 2-logic:

$$\{\langle X1;x1 \rangle,\ldots,\langle X\alpha;x\alpha \rangle,\ldots\} \vdash_2 \langle X;x \rangle$$

iff

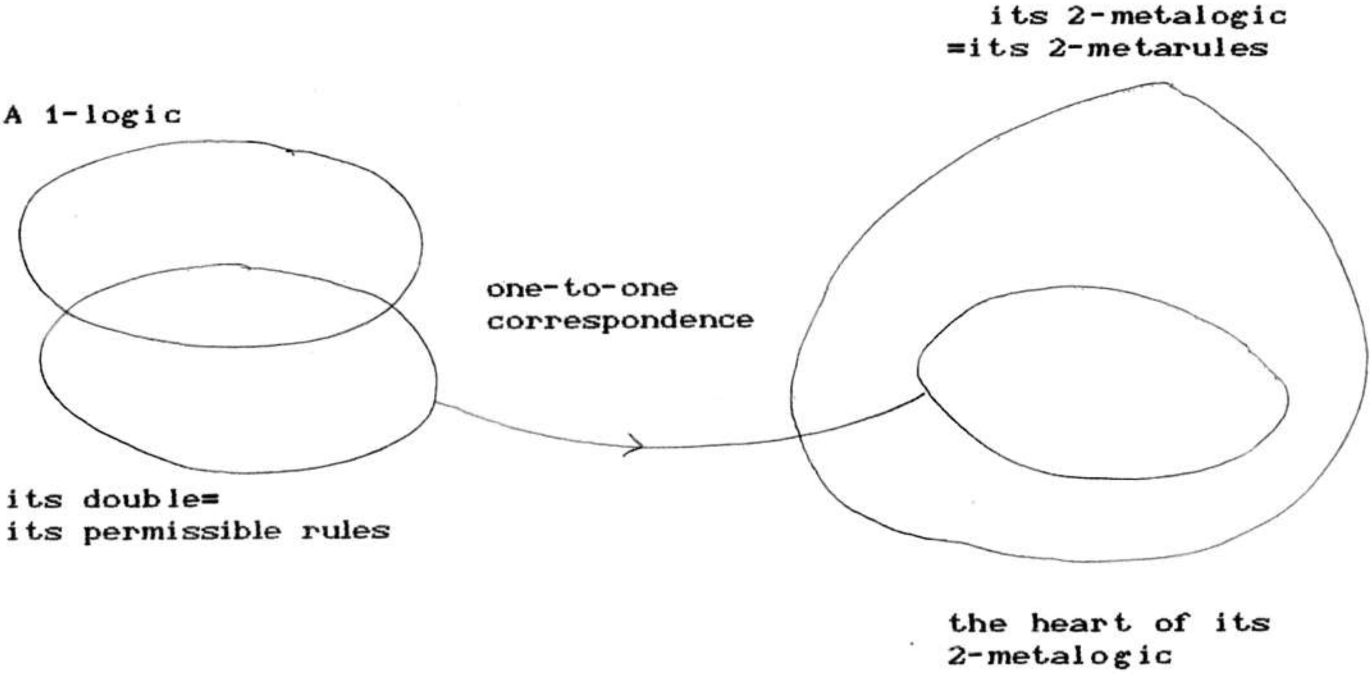
if $X1 \vdash_1 x1$ and ... and $X\alpha \vdash_1 x\alpha$ and ... then $X \vdash_1 x$.

We see that the 2-metalogic of a 1-logic is its set of 2-metarules.

A *second level metalogic* (=2-metalogic) is a second level logic which is the 2-metalogic of a first level logic.

The *heart* of a 2-metalogic is a sublogic of it defined as follows: $\{\langle X1;x1\rangle,\ldots,\langle X\alpha;x\alpha\rangle,\ldots\} \vdash \langle X;x\rangle$ iff $X1=\emptyset$, ... , $X\alpha=\emptyset$, $X=\emptyset$ and $\{\langle X1;x1\rangle,\ldots,\langle X\alpha;x\alpha\rangle,\ldots\} \vdash_2 \langle X;x\rangle$.

Thus it is clear that there is a one-to-one correspondence between the double of a logic and the heart of its 2-metalogic.



*6 ▸ *A 2-metalogic is a normal enfolding (second level) logic.*

We must not confuse this fact with the fact that the identity, cut and enfolding laws which appear as sets of 2-rules are or are not in a 2-metalogic. For example these 2-rules will be in the 2-metalogic of a logic iff it is a normal enfolding logic but the 2-metalogic of any logic will always be a normal enfolding logic.

When speaking about e.g. the cut rule we mean a set of rules, we do not make an explicit distinction here between rules and sets of rules.

*7 ▸ *Two logics are the same iff they have the same 2-metalogic.*

The semi-logic of a 2-system can be considered as a 1-logic in the following way: $\emptyset \vdash_2 \langle X;x \rangle$ iff $X \vdash_1 x$. We will say that a 1-logic is determined by a 2-system to say that it corresponds in this way to its semi-logic.

Note that a 1-logic determined by a 2-system is not necessary normal:

*8 ▸ *A object is a 1-logic iff it can be determined by a 2-system.*

Then the concept of 2-system is far more general that the one of 1-system. In fact this shows that to climb to a third level is unnecessary.

At this point it is possible to present the following picture:

if we are interested in:	we can restrict ourselves to:
1-enfolding logic	1-semi-system
1-normal logic	1-system
1-logic	2-semi-system

But the validity of this kind of restrictions is highly relative as we will see.

5. The interest of rising

The 2-metalogic of a 1-logic determined by a 2-system is not necessary the logic of this 2-system. In fact this 2-metalogic is the set of permissible rules of the 2-system and as the logic of this 2-system is its set of derived rules we have:

*9 ▸ *The 2-metalogic of the semi-logic of a 2-system is the logic of it iff it is an enfolding system.*

a 2-system $\mathfrak{R}$	derived rules of $\mathfrak{R}$ =the logic of $\mathfrak{R}$	permissibles rules of $\mathfrak{R}$ =the 2-metalogic of :
	the semi-logic of $\mathfrak{R}$	the semi-logic of $\mathfrak{R}$ considered as a 1-logic

Now we can see this very important fact: in the same way that two non-interderivable systems can determine the same semi-logic a 1-logic can be determined by two 2-systems which are not interderivable.

Using *7 we have:

*10 ▸ *If two 2-systems determine the same semi-logic they have the same permissible rules.*

But they have not necessary the same derivable rules.

*11 ▸ *If a logic can be determined by a 1-system, the identity and cut rules are permissible rules of any 2-system which determines it.*

But they are not necessary derived rules.

It is possible to study 2-systems only by interest for 1-logic and thus to consider 2-systems only as 2-semi-systems but it is clear from previous considerations that even in this case 2-logic is important because the set of derived rules of a 2-system is its logic, ie a 2-logic and the set of permissible rules of a 2-system is the 2-metalogic of its semi-logic and thus a 2-logic.

It is clear that what we call a 2-system is a simplified form of a system of type LJ of Gentzen (see [11]).

We have left aside notions of cardinality and order which are not fundamental for our present purpose and we have left open the question of the nature of basic objects.

Now it will also have been possible to consider 2-systems of type LK: in these systems constituents of rules are of type <X;Y> instead of type <X;x>. But note that considering this kind of systems does not necessary mean that we are interested in multiple-conclusion logic (see [36]): we can study 2-multiple-systems only by interest for one-conclusion logic and in the framework of one-conclusion logic ; our present abstract setting can be extended to this kind of system without rising to multiple-conclusion logic.

In fact it appears that Gentzen was using 2-ordered-multiple-systems only for the study of 1-semi-logic. The Gentzenian symbol $\rightarrow$ must not be interpreted as $\vdash$.

It is possible to say that generally the scope of logic is limited to enfolding normal logic and that we can restrict ourselves to 1-semi-systems and 1-semi-logic. But even in this treatment we are led to use more complex concepts and to climb to the second level.

All this kind of phenomena are very well known in mathematics.

One may consider for example the structure of the natural number immerged in a more complex structure only by interest for natural numbers.

But it seems that we are always drawn to more complex levels and that we are carried unwillingly to higher and higher spheres.

6. Transformations of 1-systems into 2-systems

The problem is the following: given a 1-system, how to find a 2-system whose semi-logic is the logic of this 1-system, ie how to find a 2-semi-system which determines the logic of this 1-system ?

In fact there are many ways of passing from the first to the second level so that a 1-system appears as a condensation of various 2-systems.

Straight transformation

It is clear that a 1-rule can be considered as a 2-rule: $\langle X;x\rangle=\langle\emptyset;\langle X;x\rangle\rangle$. Using symbolism:

the 1-rule: $$\frac{x1 \;\ldots\; x\alpha \;\ldots}{x}$$

is transformed into the 2-rule: $$\frac{\emptyset}{\{x1,\ldots,x\alpha,\ldots\}\succ x}.$$

Given a set $\mathfrak{R}$ of 1-rules, if we consider the 2-system $\emptyset\mathfrak{R}$ so-obtained all rules are axiomatic, then we have: $X\vdash_2 x$ iff $\langle X;x\rangle$ is in $\mathfrak{R}$. If we want more we have to put additional rules.

We consider the identity rule (id), the cut∅ rule (cut∅) and the cut rule (cut):

$$\frac{\emptyset}{X\succ x}\,\text{id}\;(x\in X)\qquad \frac{\emptyset\succ x \quad \{x\}\succ y}{\emptyset\succ y}\,\text{cut}\emptyset\qquad \frac{X\succ x \quad \{x\}\cup Y\succ y}{X\cup Y\succ y}\,\text{cut}$$

*12 ▸ *The semi-logic of a 1-system $\mathfrak{R}$ is determined by the 2-system $\emptyset\mathfrak{R}$ plus the ∅cut rule.*

*13 ▸ *The logic of a 1-system $\mathfrak{R}$ is determined by the 2-system $\emptyset\mathfrak{R}$*

plus the identity and cut rules.

Unfolding transformation

Given a 1-system $\mathfrak{R}$ we will transform it in a 2-system unf$\mathfrak{R}$ by unfolding each rule of it:

a 1-rule of $\mathfrak{R}$: $$\frac{x1 \ \ldots \ x\alpha \ \ldots}{x}$$

is transformed into the 2-rule: $$\frac{\emptyset \succ x1 \ \ldots \ \emptyset \succ x\alpha \ \ldots}{\emptyset \succ x}.$$

*14 ▸ *The semi-logic of a 1-system $\mathfrak{R}$ is determined by the 2-system unf$\mathfrak{R}$.*

If we are interested only in semi-logic this transformation is sufficient.

Now to obtain the full solution we use *extented unfolding*:

a 1-rule of $\mathfrak{R}$: $$\frac{x1 \ \ldots \ x\alpha \ \ldots}{x}$$

is transformed into the 2-rule: $$\frac{X1 \succ x1 \ \ldots \ X\alpha \succ x\alpha \ \ldots}{X \succ x}.$$

*15 ▸ *The logic of a 1-system $\mathfrak{R}$ is determined by the 2-system exunf$\mathfrak{R}$ plus the identity rule.*

It is possible that cut will be a derived rule of exunf$\mathfrak{R}$ and it is possible that it will not be. In any case cut is a permissible rule of exunf$\mathfrak{R}$: the semi-logic of eunf$\mathfrak{R}$ is the same as the logic of $\mathfrak{R}$ and the latter is a normal logic, then cut is in its 2-metalogic thus it must be in the 2-metalogic of the semi-logic of exunf$\mathfrak{R}$ which is exactly the set of permissible rules of exunf$\mathfrak{R}$.

ILLUSTRATION

1. The logic $\mathfrak{I}$ and its double

We consider here the intuitionistic implicative propositional logic $\mathfrak{I}=\langle\mathfrak{F};\vdash\rangle$.

$\mathfrak{F}$ is built from an underlying set of propositions as usual with only the connective $\to$.

We consider first this logic in itself independently of any determination, ie of any system determining it.

Recall that with only $\mathfrak{I}$ given, the following other logics are known:

$\mathfrak{D}\mathfrak{I}$, the double of $\mathfrak{I}$, ie its set of permissible rules,

$2\mathfrak{M}\mathfrak{I}$, the 2-metalogic of $\mathfrak{I}$.

For any 1-system who will determine $\mathfrak{I}$, its set of derived rules will be $\mathfrak{D}\mathfrak{I}$.

For any 2-system who will determine $\mathfrak{I}$, its set of permissible rules will be $2\mathfrak{M}\mathfrak{I}$.

The first point is that $\mathfrak{I}$ is a normal logic. Thus the identity and cut rules are in $2\mathfrak{M}\mathfrak{I}$ and they will be permissible rules of any 2-system who will determine $\mathfrak{I}$.

Now one thing is sure: $\mathfrak{I}$ is shadowy or not. But another thing is: a 2-system which determines $\mathfrak{I}$ can be shadowy or not.

Examples of permissible and non-permissible rules of $\mathfrak{I}$:

$\{p\}\vdash_{\mathfrak{D}\mathfrak{I}} p$

$\{p\}\vdash_{\mathfrak{D}\mathfrak{I}} q$

$\emptyset\vdash_{\mathfrak{D}\mathfrak{I}} (p\to p)$

$\{(p\to p)\}\nvdash_{\mathfrak{D}\mathfrak{I}} q$

$\langle\{p\};q\rangle$ is in the shadow of $\mathfrak{I}$.

Now it is time to emphasize an important point. In our abstract setting the nature of the objects was not specified. One predominant part of logic is what is called *formal logic* in the sense that an object of a formal logic can be considered as constituted by a form and by a content: formal logic is based on this distinction. In our present example the form of an object of $\mathfrak{I}$ is its set of connectives and the matter is its set of propositions (parentheses can be considered as part of the form but in fact they are subsidiary objects and can be eliminated by the famous method of Łukasiewicz). Now a substitution is a function which lets invariant the form of the object. $\mathfrak{I}$ is a formal logic:

$X \vdash x$ iff for every substitution σ $\sigma(X) \vdash \sigma(x)$.

A rule schema is a set of rules which is closed under substitution. (see [3] for a special account on substitution and formal logic).

*16 ▸ *In the shadow of* $\mathfrak{I}$ *there is no rule schema* (see [30]).

In other words: $\mathfrak{I}$ is a schema shadeless logic.

Any 1-system who will determine $\mathfrak{I}$ will be a schema shadeless logic.

2. Systems for $\mathfrak{I}$

The 1-system $\mathcal{J}$

$$\frac{x \quad (x\to y)}{y}\ \text{1-mp} \qquad \frac{\emptyset}{(x\to(y\to x))}\ \text{1-pk} \qquad \frac{\emptyset}{((x\to(y\to z))\to((x\to y)\to(x\to z)))}\ \text{1-ps}$$

Straight semi-transformation: the 2-system $\mathfrak{U}$w

$$\frac{\emptyset}{\{x,(x\to y)\} \succ y}\;\text{2-mpa} \qquad \frac{\emptyset}{\emptyset \succ (x\to(y\to x))}\;\text{2-pk}$$

$$\frac{\emptyset}{\emptyset \succ ((x\to(y\to z))\to((x\to y)\to(x\to z)))}\;\text{2-ps}$$

$$\frac{\emptyset \succ x \qquad \{x\} \succ y}{\emptyset \succ y}\;\text{cut}\emptyset$$

Straight transformation: the 2-system $\mathfrak{U}$

$$\frac{\emptyset}{\{x,(x\to y)\} \succ y}\;\text{2-mpa} \qquad \frac{\emptyset}{\emptyset \succ (x\to(y\to x))}\;\text{2-pk}$$

$$\frac{\emptyset}{\emptyset \succ ((x\to(y\to z))\to((x\to y)\to(x\to z)))}\;\text{2-ps}$$

$$\frac{\emptyset}{X \succ x}\;\text{id}\;(x\in X) \qquad \frac{X \succ x \qquad \{x\}\cup Y \succ y}{X\cup Y \succ y}\;\text{cut}$$

Unfolding semi-transformation: the 2-system $\mathfrak{S}$w

$$\frac{\emptyset \succ x \qquad \emptyset \succ (x\to y)}{\emptyset \succ y}\;\text{2-mp}\emptyset \qquad \frac{\emptyset}{\emptyset \succ (x\to(y\to x))}\;\text{2-pk}$$

$$\frac{\emptyset}{\emptyset \succ ((x\to(y\to z))\to((x\to y)\to(x\to z)))}\;\text{2-ps}$$

Unfolding transformation: the 2-system $\mathfrak{H}$

$$\frac{X \succ x \qquad Y \succ (x \to y)}{X \cup Y \succ y}\ \text{2-mp} \qquad \frac{\emptyset}{\emptyset \succ (x \to (y \to x))}\ \text{2-pk}$$

$$\frac{\emptyset}{\emptyset \succ ((x \to (y \to z)) \to ((x \to y) \to (x \to z)))}\ \text{2-ps}$$

$$\frac{\emptyset}{X \succ x}\ \text{id}_{(x \in X)}$$

By the general results of the precedent section (*12, *13, *14, *15) we know that $\mathfrak{Z}$, $\mathfrak{U}$ and $\mathfrak{H}$ determine the same 1-logic.

Now we will consider more sophisticated 2-systems.

The 2-system $\mathfrak{N}$

$$\frac{X \succ x \qquad Y \succ (x \to y)}{X \cup Y \succ y}\ \text{2-mp} \qquad \frac{X \cup \{x\} \succ y}{X \succ (x \to y)} \to\text{r}$$

$$\frac{\emptyset}{X \succ x}\ \text{id}_{(x \in X)}$$

The 2-system $\mathfrak{G}$

$$\frac{X \succ x \qquad Y \cup \{y\} \succ z}{X \cup Y \cup \{(x \to y)\} \succ z} \to\text{l} \qquad \frac{X \cup \{x\} \succ y}{X \succ (x \to y)} \to\text{r}$$

$$\frac{\emptyset}{X \succ x}\ \text{id}_{(x \in X)} \qquad \frac{X \succ x \qquad Y \cup \{x\} \succ y}{X \cup Y \succ y}\ \text{cut}$$

*17 ▸ *$\mathfrak{H}$, $\mathfrak{N}$ and $\mathfrak{G}$ are interderivable.*

The main point in this demonstration is the so-called *deduction theorem* which can be expressed as follows:

*18 ▸ →r *is a derived rule of* $\mathfrak{H}$.

Thus $\mathfrak{N}$ is derivable from $\mathfrak{H}$.

Cut is derivable from {2-mp,→r}:

$$\dfrac{X \succ x \qquad \dfrac{Y\cup\{x\} \succ y}{Y \succ (x\to y)}\,{\to}\mathrm{r}}{X\cup Y \succ y}\,\text{2-mp}$$

→l is derivable from {2-mp,cut} (modulo id):

$$\dfrac{\dfrac{\dfrac{\emptyset}{(x\to y)\succ(x\to y)}\,\text{id} \qquad X\succ x}{X\cup\{x\to y\}\succ y}\,\text{2-mp} \qquad Y\cup\{y\}\succ z}{X\cup Y\cup\{x\to y\}\succ z}\,\text{cut}$$

Thus $\mathfrak{G}$ derivable from $\mathfrak{N}$.

{2-mp} is derivable from {cut,→l} (modulo id):

$$\dfrac{X\succ x \qquad \dfrac{X\succ(x\to y)\qquad \dfrac{\dfrac{\emptyset}{\{x\}\succ x}\,\text{id}\quad \dfrac{\emptyset}{\{y\}\succ y}\,\text{id}}{\{(x\to y)\}\cup\{x\}\succ y}\,{\to}\mathrm{l}}{X\cup\{x\}\succ y}\,\text{cut}}{X\succ y}\,\text{cut}$$

{2-ps} is derivable from {→l,→g} (modulo id):

$$\dfrac{\dfrac{\emptyset}{\{x\} \succ x}\,\text{id} \qquad \dfrac{\dfrac{\dfrac{\emptyset}{\{x\} \succ x}\,\text{id} \quad \dfrac{\emptyset}{\{y\} \succ y}\,\text{id}}{\{(x\to y),x\} \succ y}\,\to l \qquad \{z\} \succ z}{\{(y\to z),(x\to y),x\} \succ z}\,\to l}{\dfrac{\{(x\to(y\to z)),(x\to y),x\} \succ z}{\emptyset \succ ((x\to(y\to z))\to((x\to y)\to(x\to z)))}\,x3}\,\to l$$

Thus $\mathfrak{H}$ is derivable from $\mathfrak{S}$.

3. The shadow of $\mathfrak{S}$-

We now consider the system $\mathfrak{S}$- which is the system $\mathfrak{S}$ without the cut rule.

A 2-rule: $\langle\{\langle X1;x1\rangle,\ldots,\langle X\alpha;x\alpha\rangle,\ldots\};\langle X;x\rangle\rangle$ is said to be *analytic* iff the xis and the members of the Xis are all subformulae of x or of formulae of X, *strictly analytic* iff moreover the proper subformulae of x and X are only subformulae of the xis and the Xis.

If we have a set of analytic rules $\mathfrak{R}$ it is clear that all the derived rules of $\mathfrak{R}$ will be analytic.

All the rules in $\mathfrak{S}$- are analytic, thus:

*19 ▸ *The cut rule is not a derivable rule of* $\mathfrak{S}$-.

*20 ▸ $\mathfrak{S}$ *and* $\mathfrak{S}$- *determine the same semi-logic* (Corollary of the cut-elimination theorem).

Now it is clear that the cut rule is a permissible rule of $\mathfrak{S}$- (cf *11, *17 and *20). Thus

*21 ▸ $\mathfrak{S}$- *is a schema shadowy logic.*

We see that $\mathfrak{J}$ is determined by two non-interderivable 2-systems.

In fact all non-analytic permissible rules of $\mathfrak{G}-$ are non-derivable rules, for example 2-mp. The question is: are all the rules in the schema shadow of $\mathfrak{G}-$ non-analytic ?

The answer is negative.

Consider the following rule schema:

monotony rule:

$$\frac{X \succ x}{Y \succ y} \quad (X \subseteq Y).$$

The system $\mathfrak{G}-$ is often presented with this additional rule.

*22 ▸ *The monotony rule is a permissible rule of $\mathfrak{G}-$.*

It is clear that this rule is in the 2-metalogic of any 1-system, and in particular it is in the one of $\mathfrak{J}$, now because $\mathfrak{G}-$ determines the same 1-logic as $\mathfrak{J}$ (deduction theorem/ cut elimination theorem) it is by definition a permissible rule of $\mathfrak{G}-$.

Using *4, we see that if we add this rule to $\mathfrak{G}-$ we obtain the same semi-logic.

*23 ▸ *The monotony rule is not a derivable rule of $\mathfrak{G}-$.*

Now all the rules in $\mathfrak{G}-$ are *strictly analytic* and thus all derivable rules of $\mathfrak{G}-$ are strictly analytic and the monotony rule is not.

From *22 and *23 we infer that:

*24 ▸ *The monotony rule is in the schema shadow of $\mathfrak{G}-$.*

4. About the classification of systems

An ordinary classification of systems of deduction is (see e.g. [37]):

- Hilbertian/axiomatic

- Natural deduction
- Sequent calculus.

In fact all these systems can be considered as systems of the same type: 2-systems, in our examples, respectively: ℌ, 𝔑, 𝔊/𝔊⁻.

Now among 2-systems we can consider various types of systems according to the type of their rules.

Following Gentzen (see [11]) rules can be divided in two classes: structural and non-structural (=operational) rules.

We have the following picture of the repartition of rules:

𝔑

	non-structural	structural
non-axiomatic	2mp →r	∅
axiomatic	∅	id

ℌ

	non-structural	structural
non-axiomatic	2mp	∅
axiomatic	pk ps	id

𝔊

	non-structural	structural
non-axiomatic	→l →r	cut
axiomatic	∅	id

𝔘

	non-structural	structural
non-axiomatic	∅	cut
axiomatic	pks mpa	id

In 𝔘 every step in a proof is produced by the cut rule. Such a system will be called a *barbarian system*: it is of a high interest both from a philosophical and mathematical point of view (see [2]).

5. Modus ponens: What is it and how to write it

According section 2 above the *modus ponens* can be considered as 1-mp, 2-mpa, 2-mp∅ or 2-mp.

The first point is that we must not use in all these symbolisations $\vdash$ instead of $\succ$ because a rule is not an assertion (see [2] for discussion).

The second point is that we must use the right formulation into the right framework.

To answer the question: what is the modus ponens ? We must first state in which framework we are working.

Historical Note

1. Confusion between 1-concepts and 2-concepts

The usual presentation of what is called an Hilbertian system runs like this:

a semi-logic is defined by an inductive definition: a set of theses is generated from a set of axioms by rules.

This system can be interpreted as a 1-system or as a 2-system, in our examples as $\mathcal{J}$ or as $\mathfrak{H}$w.

When this definition is extended as to go from semi-logic to logic the ambiguity is still there, in our examples: $\mathcal{J}$ or $\mathfrak{H}$.

In [19] (Kleene) these systems are considered as 1-systems ([19] p.83, p.87 and p.88) but G.Sundhom [37] considers them rather as 2-systems.

It is not clear at all in this context whether a rule is a 1-rule or a 2-rule. Kleene considers rules as 1-rule ([19] pp.82-83) but what he calls a derived rule ([19] p.86) is a 2-rule. In the same way what Church calls a derived rule is a 2-rule ([4] p.83, p.93, p.94, p.165). In fact derived rules are considered by Kleene and Church as metamathematical theorems ([19] p.86, [4] p.83). Derived rules are here what we have called 2-metalogical rules, ie 2-permissible rules and there are not distinguished from 2-derived rules which are not explicitly considered.

For example the deduction theorem is considered as a metatheorem (a derived rule). But if we are working really in 1-systems, this is not a derived rule. Dummet is misleading when saying that the deduction theorem states that $\rightarrow+$ $(\Rightarrow\rightarrow r)$is a derived rule of his 1-system Ax ([8] p.127). In our examples: the deduction theorem is a derivable rule of $\mathfrak{H}$ but not of $\mathfrak{Z}$. It is a 2-metalogical rule of $\mathfrak{J}$.

2. Permissibility

The notion of permissible rule was putting forward at the end of the fifties by P.Lorenzen [21], Moh Shaw-Kwey [35], Hiż [17] and later studied by K.Schütte [32], H.Wang [42], H.C.Wasserman [43], W.A. Pogorzelski [26].

What was emphasized is that the set of permissible rules is not stable by extension of systems: a permissible rules can ceased to be permissible if we extend the system. But note that the converse is false: a permissible rule can be permissible in all the extensions of a system, this is the case of the cut rule in $\mathfrak{G}$- (if we consider extensions in the natural way: by schema of rule).

The concept of permissible rule appeared as a strange phenomenon mainly because of things described in our remark following *5.

Althought, as we have seen in the precedent section, the distinction between 1-systems and 2-systems is rather confused it seems that the concept of permissible rule was developed essentially in the framework of 1-concepts. The fact that the cut rule is a 2-permissible and not a 2-derived rule was never stated clearly.

J.Porte [29] was the first to make an explicit distinction between systems ('systèmes logistiques', but such systems are considered independently of the logic they determine), semi-logic (systèmes thétiques) and logic (systèmes déductionnels) and to study their relations. But the distinction between 1-systems and 2-systems is not explicit. He gives the following definitions ([29] p.35): a primitive rule R is T-independent from a set of rules $\mathfrak{R}$ iff the semi-logic of $\mathfrak{R}-\{R\}$ is the same as the semi-logic of $\mathfrak{R}$ and a primitive rule is D-independent iff the logic of $\mathfrak{R}-\{R\}$ is the same as the logic of $\mathfrak{R}$. If we interpret these definitions for 2-rules and 2-systems, using this terminology it is clear that in $\mathfrak{U}w$ the identity rule is T-independent but not D-independent. But what is strange is that Porte says ([29] p.37) that the cut rule is D-independent and not T-independent, but in $\mathfrak{S}$- it is T-independent and not D-independent.

The fact that the status of the cut rule is not clear appears for example in an article by D.S.Scott where he says: "In many formalizations a great deal of effort is expended to eliminate cut as a primitive rule ; but it has to be proved as a derived rule. In general, cut is *not* eliminable (...). It is only for some very special relations (...) that the rule can be avoided as an assumption." ([35] p.414).

We must emphasize the following points. The cut rule is not a derived rule of the system $\mathfrak{G}$-. It is very easy to find a system for $\mathfrak{I}$ where cut is not a primitive rule but is a derived rule, e.g. $\mathfrak{N}$ and $\mathfrak{H}$. To prove in such a system that cut is a derived rule has nothing to do with the cut-elimination theorem of Gentzen.

3. On the interpretation of Gentzen's symbol →

As it is known Gentzen writes a sequent with →. But the signification of this symbol is not really clear. Obviously Gentzen took this symbol from Hertz (see Gentzen [10] 1932). We must recall that Whitehead and Russel use the symbol ⊃ for implication and also as a kind of meta-implication ([44] 1910). Hilbert uses the symbol → for implication ([14] 1922). In this context the use by Hertz of the symbol → is not clear at all, though he himself says that he uses it as the formal implication of Whitehead and Russell ([13] p.247 footnote one, 1922). It is clear that when Gentzen uses this Hertzenian symbol, it is not formal implication, in fact he uses ⊃ for formal implication. We have introduced here the symbol ≻ because we are using → for implication. In any case it is absurd to use ⊢ for the Gentzenian symbol → but it appears that it is useful to use a more suggestive symbol than ';' or ':' as in [8]. The symbol ≻ seems good because it suggests a connection with ⊢, but the left part is not the vertical stroke used by Frege ([9] 1879) to symbolize the assertional mood. (For a special treatment of this subject see [2]).

The way Gentzen writes rules permits directly the interpretation that we have made here, ie the distinction between two strictly parallel systems (1-systems and 2-systems), and which was in fact developed by G.C.Moisil which considered a hierarchy of calculi according to the complexity of the underlying objects [23][24].

4. On the theory of the consequence operator

At the end of the twenties, Tarski introduced the consequence operator Cn [39]. Obviously this is a general treatment of 1-logic. But not of 1-systems. The idea of Tarski was to provide an axiomatisation of the notion of deduction as it appears in every deductive theory [40][41]. In fact this is an axiomatisation of 1-systems of deduction as we can see by the chosen axioms for the consequence operator. The axiomatisation of the deduction can be viewed as a system of deduction but in fact Tarski never used the concept of rule in this context (see [1] [7] for commentaries). The notion of rule was given for the first time in the framework of the theory of the consequence operator by Łos and Suszko in 1958 [22]. This definition is a very general definition of 1-rule which is quite similar to the one presented here, but they define the notion of rule on a set $\mathfrak{X}$ which is taken as an absolute free algebra, a vision of the ordinary set of propositional formulae which permits to give a definition of schema in terms of endomorphisms, what they call a structural rule (this word is used in a different way by Gentzen, see sect2, part2). Although Suszko [38] was aware of the discoveries of Hiż, the notion of permissibility was directly treated only later by W.A.Pogorzelski [26] who has defined the notion of derivable and permissible rules inspired by H.Wang [42]. He has introduced the notion of structural completeness of a system [26]: every structural permissible rule is derivable. In this context structural completeness has been systematically studied [28] and for example T.Prucnal [30] has proved the structural completeness of the implicative intuitionistic propositional logic (cf *16). But all this takes place in the context of 1-concepts. To study 2-systems in the context of the structural theory of consequence operator one need to consider the basic set as a more complex algebra, with two types of functions. In fact 2-systems were studied only later in this context [45] and the notion of 2-derivability, 2-permissibility and 2-structural completeness have not been studied yet in the framework of the theory of the consequence operator.

Our present exposition was inspired by several works of N.C.A. da Costa (see e.g. [12][20]) who has defined the notion of 1-system and 1-rule in a very general way. A first development of abstract definitions of 1-system and 1-rule as well as of 2-system and 2-rule has been presented by N.C.A da Costa and the author in [5].

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This paper has been written when the author was at the Department of Logic of the University of Wrocław (Poland) in the setting of Franco-Polish cooperation (French Foreign Office/ Polish Ministry of Education).

The author would like to thank the Professor Jan Zygmunt for commentaries, informations and bibliographical supply.

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