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Paraconsistent logic! (A reply to Slater)

PARACONSISTENT LOGIC! (A REPLY TO SLATER)

Jean-Yves Béziau*

Je ne discute jamais du nom pourvu qu'on m'avertisse quel sens on lui donne.

Blaise Pascal, *Les Provinciales*

Contents

0. Paraconsistent logic

Paraconsistent logic is the study of logics in which there are some theories embodying contradictions but which are not trivial, in particular in a paraconsistent logic, the *ex contradictione sequitur quod libet*, which can be formalized as $Cn(T, a,\neg a)=\mathbf{F}$ is not valid. Since nearly half a century various systems of paraconsistent logic have been proposed and studied. This field of research is classified under a special section (B53) in the *Mathematical Reviews* and watching this section, it is possible to see that the number of papers devoted to paraconsistent logic is each time greater and has recently increased due in particular to its applications to computer sciences (see e.g. Blair and Subrahmanian, 1989).

However in a recent paper entitled «Paraconsistent logics?», a philosopher from Perth, B.H.Slater, pretends to show in less than ten lines that paraconsistent logic doesn't exist. Here is his laconic argument:

> If we called what is now «red», «blue», and vice versa, would that show that pillar boxes are blue, and the sea is red? Surely the facts wouldn't change, only the mode of expression of them. Likewise, if we called «subcontraries», «contradictories», would that show that «it's not red» and «it's not blue» were contradictories? Surely the same point holds. And that point shows that there is no «paraconsistent logic». (Slater 1995, p.451)

Are these few lines, the death sentence of paraconsistent logic?

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Slater' argumentation is based on the traditional notions of «contradictories» and «subcontraries». Unfortunately the Perthian doesn't give precise definitions of them. After giving such definitions and proving a general result about them, we will show that Slater's argument is not valid or, in the best case, is tautological.

1. Contradictories, subcontraries and contraries in the tradition

Such notions as «subcontraries» and «contradictories» belong to traditional logic, i.e. logic in the tradition of Aristotle. The first point is to precise what is their meaning in this tradition and the second point is to see how they can be understood in the light of modern mathematical logic.

One of the sad defect of Slater's argument is that both of these points are eluded and that therefore his argument is viciated by fuzziness. The farther precision Slater is getting at is when he says that *contradictories* cannot be true together - by definition» (Slater 1995, p.453). Even this precision is quite ambiguous because, due to the fact the Perthian doesn't give any definition of contradictories, one may imagine that the definition of contradictories is that two sentences are contradictories iff they cannot be true together, which is not the correct definition according to the tradition as we shall see very soon.

Of course one can imagine that it is not necessary to precise what is the exact meaning of notions such as contradictories and subcontraries, that everybody knows what their meaning is, and that this meaning is clear. But it is not so obvious, due to the fact that these notions belong to traditional logic, and that most concepts of traditional logic appear as confuse in the light of modern logic, and that at least their interpretations is not straightforward.

We will not enter into philological details to explain what is the meaning of «contradictories», and «subcontraries». The following excerpt from p.56 of (Kneale and Kneale 1962) will provide all the necessary information for our discussion including the standard definitions of contradictories, subcontraries and contraries (the concept of subalterns is not relevant for us here):

> … the square of opposition, is also not to be found in Aristotle's text, but it provides a useful summary of his doctrine. According to his explanations, statements are opposed as *contradictories* when they cannot both be true and cannot both be false, but as *contraries* only when they cannot both be true but may both be false [*De Interpretatione* 7 (17b 16-25)] … Although he does not use these expressions *subaltern* and *sub-contrary*), Aristotle (…) assumes that subcontraries cannot be false though they may both be true. This is shown by his description of them as contradictories of contraries.

For more details about the square of opposition, the reader may consult e.g. (Parsons 1997).

3. Contradictories, subcontraries and contraries in classical logic

Let **F** be the set of propositional formulas built with the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$. Formulas will be denoted by a, b, etc., sets of formulas by T, U, etc. The set **C** of classical valuations is defined as usual: it is a set of functions from **F** to $\{0,1\}$ and its members obey the standard conditions, in particular we have: for any v in **C** and for any a in **F**, $v(a)=1$ iff $v(\neg a)=0$.

With this framework we are now able to define precisely the discussed notions in the context of the semantics of classical logic.

Given two formulas a and b, we say that they are:

- contradictories iff for any v in **C**, $v(a)$=0 iff $v(b)$=1;
- contraries iff for any v in **C**, $v(a)$=0 or $v(b)$=0 and there exists v in **C**, $v(a)$=0 and $v(b)$=0;
- subcontraries iff for any v in **C**, $v(a)$=1 or $v(b)$=1, and there exists v in **C**, $v(a)$=1 and $v(b)$=1;

Let us note that if we remove the second part of the definition of subcontraries

«there exists v in **C**, $v(a)$=1 and $v(b)$=1», which translates «may both be true», then all contradictories are subcontraries. In this case confusing subcontraries with contradictories would not be the same as switching red with blue, or cats with dogs, but rather would amount of confusing dogs with canines. Let us call *global confusion* this kind of error by contrast to the first one that we can call *switching confusion*. As Slater claims through his red and blue example that paraconsistent logicians are making a switching confusion rather than a global one, it seems implicit that he doesn't consider that all contradictories are subcontraries, neither do we here.

It is clear that for any formula a, a and $\neg a$ are contradictories. The connective $\neg$ is said to be a *contradictory forming relation*.

Which examples of subcontraries can we find? For any two atomic formulas a and b, a and $\neg a \vee b$ are subcontaries, as the reader can easily check. This can be illustrated by «Plato is a cat» and «Plato is not a cat or snow is blue», which cannot both be false but can both be true.

Can we define the relation which associates to any formula a the set of formulas $\{\neg a \vee b; b \in \mathbf{F}\}$ as a *subcontrary forming relation*? That sounds reasonable but we must be aware that in this case this relation includes pairs of formulas like a and $\neg a \vee (a \wedge \neg a)$ which are contradictories.

It is clear that inside classical logic, there are a lot of subcontrary forming relations; however the question is: are paraconsistent negations part of these subcontrary forming relations? And the answer is: no. Because these negations are not definable in classical logic.

For example da Costa's paraconsistent negation of the logic C1 is not definable in classical logic because it is not self-extensional (i.e. the replacement theorem does not hold for it).

A paraconsistent negation is not in general a subcontrary forming relation inside classical logic, maybe be it is a subcontrary forming relation from another point of view - this question will be examined later on - but anyway we must remember that in general paraconsistent negations are not definable in classical logic and that for example the logic C1 of da Costa is *strictly stronger* than classical logic in the sense that classical logic is definable in C1 but not the converse. The same happens with intuitionistic logic, and that is why from this point of view, intuitionistic negation is not a contrary forming relation, erroneous conclusion that someone may reach applying an argument similar to Slater's one.

Thus paraconsistent logic is not merely the result of changing the names of concepts of classical logic already existing, but the appearance of a new phenomenon. This is a first point against Slater.

Even if someone thinks that notions such as negation and contradictory cannot be used in another way that the way they are used in classical logic, he must admit that there are notions of non classical logic that cannot be defined in classical logic (and that therefore, however they are named, these notions cannot be named by names naming some notions definable in classical logic).

As I have pointed out in my review of Slater's paper for *Mathematical Reviews* (96e03035), paraconsistent logic is not a result of a verbal confusion similar to the one according to which in Euclidean geometry «point» will be exchanged with «line», but rather the shift of meaning of «negation» in paraconsistent logic is comparable to the shift of meaning of «line» in non-Euclidean geometry.

3. Contradictories, subcontraries and contraries in paraconsistent logic

31. Da Costa's logic C1

The set of formulas of the logic C1 is the same set of formulas of classical logic. This logic was presented syntactically in (Costa 1963) and its semantics presented in (Costa 1976).

The semantics for C1 is a non truth-functional semantics. Its set **D** of bivaluations can be defined like this: $v \in \mathbf{D}$ iff v is a function from **F** into {0, 1} obeying the following conditions:

- if $v(a)=0$ then $v(\neg a)=1$

- if $v(a \wedge \neg a)=1$, the $v(\neg (a \wedge \neg a))=0$

- if $v(a)=0$, then $v(\neg ?\ a)=0$

- if $v(a\#b)=1$ and $v(a) \neq v(\neg a)$ and $v(b) \neq v(\neg b)$,

then $v(\neg(a\#b))=0$, where $\# \in \{\wedge, \vee, \rightarrow\}$.

These are the conditions for negation. We will not recall the conditions for the other connectives which are similar to the classical case (note however that the semantics for C1 cannot be generated by distributions on atomic formulas as it is the case in classical logic or other truth-functional semantics).

It is clear that if we redefine the notion of *contradictories*, *subcontraries*, and *contraries* inside C1 (i.e. using **D** instead of **C** in the definition of SECTION 2), then the paraconsistent negation ¬ of C1 is not a contradictory forming relation but is a subcontrary forming relation.

It is worth mentioning that da Costa has also developed a logic in which there is a paraconsistent negation which is neither a contradictory forming relation, nor a sucontrary forming relation, nor a contrary forming relation, from the point of view of the set of valuations of this logic (Loparic and Costa 1984).

32. Priest's logic LP

Priest has proposed a rival system to da Costa's one called LP (logic of paradox), presented for the first time in (Priest 1979). Priest claims that his logic is better than da Costa's, in particular because, according to him and Routley, da Costa's paraconsistent negation is not a negation but a subcontrary forming relation.

The argumentation of Priest and Routley appears in (Priest and Routley 1989). In the same paper the two pseudo-Australian claim that their argumentation against C1 cannot be applied to LP:

> Someone might try to make out that the negation of this system is not really a negation. But in virtue of all the above points, they would have little ground to stand on. (Priest and Routley 1989, p.169)

However Slater in his paper attacks also Priest's logic and says that the paraconsistent negation of Priest is also only a subcontrary forming relation. Although the argumentation of the Perthian is quite imprecise, and in particular is false in the sense that Priest's negation is not a subcontrary forming relation inside classical logic, it contains a valid remark that we will try to make clear.

Priest's semantics for his logic LP can be presented in different manners. It can be seen as a three-valued (truth-functional) semantics. The set of valuations **P** is a set of functions from **F** to {0, ½, 1}, obeying the following conditions for negation: for any v in **P**, and any a in **F**,

- $v(a)=0$ iff $v(\neg a)=1$
- $v(a)= ½$ iff $v(\neg a)= ½$.

Now if we want to interpret the discussed traditional notions in this context (more generally in the context of a logic with more than two values), we must fix what «truth» is and what «falsity» is. It is clear that if we interpret truth by 1 and falsity by 0, then ¬ is a contradictory forming relation. And that is apparently why Priest thinks that his paraconsistent negation is really a negation. But his argumentation is viciated as Slater himself confusedly perceived.

The reason why Priest's argumentation is wrong is the following: he considers as designated elements (in the sense of matrix theory) not only 1 but also ½, as we can see when he defines the notions of logical truth and semantic consequence. The last one is defined by:

$$a\in Cn(T) \text{ iff for every } v\in \mathbf{P},\ v(b)=0 \text{ for one } b\in T, \text{ or, } v(a)=1 \text{ or } v(a)= ½.$$

This definition allows to have $a\notin Cn(b,\neg b)$, for any atomic formulas a and b, and therefore to say that LP is paraconsistent. Had 1 been taken as the only designated value, LP would have not been paraconsistent.

Priest's conjuring trick is the following: on the one hand he takes truth to be only 1 in order to say that his negation is a *contradictory forming relation*, and on the other hand he takes truth to be ½ and 1 to define LP as a paraconsistent logic. However it is reasonable to demand to someone to keep his notion of truth constant, whatever it is. Therefore we have only the two following possibilities, which show that Priest cannot run away: in one case LP is paraconsistent and its negation is only a subcontrary forming relation from the point of view of **P**, in the other case LP's negation is a contradictory forming relation but LP is not paraconsistent.

We cannot have the penny and the bun, that is what we will show explicitly in the next section.

4. A general result about contradictories and paraconsistent logic

It seems to us that the real question is to know whether a paraconsistent negation can be a contradictory forming relation from the point of view of its own semantics. We have seen that it is neither the case of da Costa's negation, nor of Priest's negation. In this section we will show that in general it is not possible for a paraconsistent negation to be a contradictory forming relation from the point of view of its own semantics.

For proving this result we will have to discuss and present succintly some general remarks on logic and semantics. This will permit us by the way to precise some points made about Priest's logic.

The notion of contradictories depends on the notions of truth and falsity. One may think that in the case of many-valued logics, the notion of contradictories would therefore be seriously challenged. But following the traditional matrix approach to many-valued logic, it is not really challenged because fundamentally a bivalent division is kept, as stressed by G.Malinowski:

> The matrix method inspired by truth-tables embodies a distinct shadow of two-valuedness in the division of the matrix universe into two subsets of designated and undesignated elements. (Malinowski 1993, p.72)

What happens is that matrices are used to define logical truth and also consequence relation in a way that there is no doubt that designated values should be taken as truth and undesignated values as falsity. Of course it would be possible to use many-valued matrices in a more radical way, breaking the bivalent paradigm, as proposed in (Malinowski 1994), but this is not what is done generally and in particular this is not what Priest is doing, as we have seen.

GENERAL DEFINITION OF CONTRADICTORIES

The notion of contradictories can be defined for any set of bivaluations **B** on a given set **L**, i.e. when **B** is a set of functions from **L** to {0,1}:

Given two objects *x* and *y* of **L**, we say that *x* and *y* are *contradictories* iff for every $v \in B$, $v(x)=0$ iff $v(y)=1$.

DEFINITION OF LOGIC

We call a logic ***L*** any structure ***L***= <**L**;*Cn*> where **L** is any set and *Cn* any function from the power set of L into itself.

Remark We therefore do not presuppose that *Cn* obeys any axiom, or that L is a structure of a particular kind. Our reasoning can thus be applied to any logical language.

DEFINITION OF CLASSICAL NEGATION

Given a logic ***L***=<**L**;*Cn*>, a unary function ¬ on L is said to be a *classical negation* iff for every $x \in \mathbf{L}$ and $T \subseteq \mathbf{L}$,

$$x \in Cn(T) \text{ iff } Cn(T, \neg x) = \mathbf{L}$$

This definition is equivalent to other standard definitions of classical negation (see Béziau 1994).

We can ask: is classical negation a contradictory forming relation (i.e. a relation such that for every *x*, *x* and ¬*x* are contradictories)? But contradictories in which sense?

Contradictories from the point of view of any set of bivaluations which can define the logic of this negation, i.e. any adequate bivalent semantics for this logic. Before turning to the definition of adequate bivalent semantics, let us note that therefore the notion of contradictory here makes sense only if the logic can be defined by a set of bivaluations. This is the case of a wide class of logics, including most of many-valued logics (on this topic see Costa and Béziau 1994).

Note also that the theorem we will prove below makes sense only if we are in the case of logics which can be defined by a set of bivaluations, but that the proof of the theorem does not depend on any specific axioms for Cn.

DEFINITION OF ADEQUATE BIVALENT SEMANTICS

Given a logic $\boldsymbol{L}$=<$\mathbf{L}$;Cn>, a set of functions $\mathbf{B}$ from $\mathbf{L}$ to {0, 1} is called an *adequate bivalent semantics* iff for every $x \in \mathbf{L}$ and $T \subseteq \mathbf{L}$:

$$x \in Cn(T) \text{ iff for every } v \in \mathbf{B}, \text{ if } v(y)=1 \text{ for every } y \in T \text{ then } v(x)=1.$$

THEOREM

$\neg$ is a classical negation (in a given logic $\boldsymbol{L}$) if for every x, x and are contradictories (from the point of view of any adequate bivalent semantics for $\boldsymbol{L}$).

Proof. Suppose that for every x, x and $\neg x$ are contradictories and that $\neg$ is not a classical negation.

1) There exists x, T and y, such that $x \in Cn(T)$ and $y \notin Cn(T,\neg x)$. If $y \notin Cn(T,\neg x)$, then there exists v, such that $v(T)$=1, $v(\neg x)$=1, $v(y)$=0. But if $x \in Cn(T)$ and $v(T)$=1, then $v(x)$=1. Therefore x and $\neg x$ are not contradictories, because they can both be true.

2) There exists x and T such that $x \notin Cn(T)$ and $Cn(T,\neg x)=\mathbf{L}$. If $x \notin Cn(T)$, then there exists v such that $v(T)$=1 and $v(x)$=0. Now suppose that $v(\neg x)$=1, then $x \notin Cn(T,\neg x)$, which is absurd due to the fact that $Cn(T,\neg x)=\mathbf{L}$. Therefore $v(\neg x)$=0. Therefore x and $\neg x$ are not contradictories, because they can both be false.

Remark The converse of this theorem is false. It can be proved (with some few additional negligible hypotheses) that if $\neg$ is a classical negation, then for every x, x and $\neg x$ are contraries, but it cannot be proved that x and $\neg x$ are subcontraries. One counter example is the following: as a corollary of a general result, the set of characteristic functions of deductively closed sets of formulas is an adequate bivalent semantics for classical logic. But it is clear that given two atomic formulas a and b, $a \notin Cn(b)$ and $\neg a \notin Cn(b)$.

COROLLARY

Given a paraconsistent negation $\neg$ (in a logic $\mathbf{L}$), x and $\neg x$ cannot be contradictories for every x (from the point of view of any adequate bivalent semantics for $\mathbf{L}$)}.

In another words: a paraconsistent negation cannot be a contradictory forming relation from the point of view of its own semantics (and the same holds of course for intuitionistic negation, Curry's negation, Johansson's negation, etc.).

We have not given a precise definition of paraconsistent negation, and in fact there is no uniform definition, but to infer the COROLLARY from the THEOREM, we just need to suppose that a paraconsistent negation is different from classical negation. So if we consider the rejection of the *ex contradictione sequitur quod libet*, $Cn(T, a, \neg a) = \mathbf{F}$, as a necessary condition for a negation to be paraconsistent, it is enough to get the COROLLARY.

5. Conclusion

In view of the above result, to say that a negation is not a negation because it is not a contradictory forming relation, is just to say that a negation is not a negation because it is not a classical negation, because only classical negation is a contradictory forming relation.

To state, without argumentation, that only classical negation is a negation and to claim that paraconsistent negations are therefore not negations, is just to make a tautological affirmation without any philosophical value.

But the real discussion does not reduce to such a trivial point. The question is to know what are the properties of classical negation which are compatible with the rejection of the *ex contradictione sequitur quodlibet*, rejection which is the basis of paraconsistent negation (on this topic see Béziau 2000).

Paraconsistent logic has shown in fact that a paraconsistent «negation» can have some strong properties, that for example it does not reduce to a mere modal operator and that it can make sense to use the word «negation» in the context of paraconsistency, in a similar way that it can make sense to speak of «intuitionistic negation» or of «Johansson's negation».

Moreover, obviously the meaning of the word «negation» in natural language does not reduce to the meaning of classical negation of classical logic and nobody has yet tried to prohibit the use of this word in natural language.

Finally, a possible way to consider that a paraconsistent negation (or another non classical negation) is a contradictory forming relation, despite of our negative result of SECTION 4, is to change the definition of contradictory forming relation and to say that two formulas *a* and *b* are contradictories iff one is the «negation» of the other.

Of course this can lead to nonsense if we are dealing with something which has nothing to do with negation. But if we reasonably change the meaning of «negation», it makes sense to accordingly change the meaning of «contradictories».

It seems that this is the option Priest has now taken after we present to him our present criticisms to his paper with Routley.

It is worth emphasized that from this point of view Priest's negation LP does not present any superiority to da Costa's negation C1 or other paraconsistent negations.

Postface

This paper was originally written in 1996, just after I wrote the review of Slater's paper, «Paraconsistent logics?» for *Mathematical Reviews*; a Romanian translation of it was published in 2004 in I.Lucica et al. (eds), *Ex falso qodlibet*, Tehnica, Bucarest. In particular this paper was written before the publication of Greg Restall's paper, «Paraconsistent logics!», *Bulletin of the Section of Logic* 26/3 (1997), with a title which is quite the same. However the contents of the papers are completely different. After writing this paper I wrote several papers which are a continuation of it:

J.-Y.Béziau, «Paraconsistent logic from a modal viewpoint», *Journal of Applied Logic*, 3 (2005), pp.7-14. [<http://www.unine.ch/unilog/jyb/jyb-wopalo-elsevier.pdf>]

J.-Y. Béziau, «New light on the square of oppositions and its nameless corner», *Logical Investigations*, 10, (2003), pp.218-232. [<http://www.unine.ch/unilog/jyb/sep.pdf>]

J.-Y.Béziau, «Are paraconsistent negations negations?», in *Paraconsistency: the logical way to the inconsistent*, W.Carnielli et al. (eds), Marcel Dekker, New-York, 2002, pp.465-486.

6. Bibliography

[Bez94] J.-Y.Béziau, 1994, «Théorie législative de la négation pure», *Logique et Analyse*, **147-148**, pp.209-225.

[Bez96] J.-Y.Béziau, 1996, review of [Sla], *Mathematical Reviews*, 96e03035.

[Bez97] J.-Y.Béziau, 1997, «What is many-valued logic?», in *Proceedings of the 27th International Symposium on Multiple-Valued logic*, IEEE Computer Society, Los Alamitos, pp.117-121.

[Bez00] J.-Y.Béziau, 2000, «What is paraconsistent logic?», in *Frontiers in paraconsistent logic*, D.Batens et al. (eds), Research Studies Press, Baldock, pp.95-111.

[BlS] H.A.Blair and V.S.Subrahmanian, 1989, «Paraconsistent logic programming», *Theoretical Computer Science*, **68**, pp.135-153.

[Cos] N.C.A.da Costa, 1963, «Calculs propositionnels pour les systèmes formels inconsistants», *Comptes Rendus de l'Académie des Sciences de Paris*, **257**, pp.3790-3793.

[CoA] N.C.A.da Costa and E.H.Alves, 1976, «Une sémantique pour le calcul C_1», *Comptes Rendus de l'Académie des Sciences de Paris*, **238A**, pp.729-731.

[CoB] N.C.A.da Costa and J.-Y.Béziau, 1994, «Théorie de la valuation», *Logique et Analyse*, **146**, pp.95-117.

[KnK] W.Kneale and M.Kneale, 1962, *The development of logic*, Clarendon Press, Oxford.

[LoC] A.Loparic and N.C.A.da Costa, 1984, «Paraconsistency, paracompleteness and valuations», *Logique et Analyse*, **106**, pp.119-131.

[Mal] G.Malinowski, 1993, *Many-valued logics*, Clarendon Press, Oxford.

[Mali] G.Malinowski, 1994, «Inferential many-valuedness», in *Philosophical logic in Poland*, J.Wolenski (ed), Kluwer, Dordrecht, pp.75-84.

[Par] T.Parsons, 1997, «The traditional square of opposition», *Stanford Encyclopedia of Philosophy*, http://plato.stanford.edu.

[Pri] G.Priest, 1979, «Logic of paradox», *Journal of Philosophical Logic*, **8**, pp.219-241.

[PrR] G.Priest and R.Routley, 1989, «Systems of paraconsistent logic», in *Paraconsistent logic: Essays on the inconsistent*, Philosophia, Munich, pp.151-186.

[Sla] B.H.Slater, 1995, «Paraconsistent logics?», *Journal of Philosophical logic*, **24**, pp.451-454.

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Jean-Yves Béziau
Institute of Logic and Semiological Research Center
University of Neuchâtel, Espace Louis Agassiz 1
CH - 2000 Neuchâtel, Switzerland
<jean-yves.beziau@unine.ch>