Are paraconsistent negations negations? *

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Dedicated to Prof. Newton C.A. da Costa for his 70th birthday

Abstract
To know if paraconsistent negations are negations is a fundamental issue: if they are not, paraconsistent logic does not properly exist. In a first part we present a philosophical discussion about the existence of paraconsistent logic and the surrounding confusion about the emergence of possible paraconsistent negations. In a second part we have a critical look at the main paraconsistent negations as they appear in the literature.

Contents

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Does paraconsistent logic exist?

The principle of non-contradiction can be expressed in many different ways (not necessarily equivalent). One of them is: *a proposition and its negation cannot be true together*. Because the principle of non-contradiction is generally admitted, if someone says for example: "It is raining and it is not raining", this seems quite absurd.

A paraconsistent logic is a logic in which there is a negation, a *paraconsistent* negation, which does not obey the principle of non-contradiction. Such an entity may appear as a paradoxical funny object, like a plane that does not fly, or champagne without alcohol. Even worse: one may think that such an object is a contradictory thing, an impossible object, like a round square or a rocket which goes faster than the speed of light.

In fact, in a recent paper, B.H.Slater [1] claims that paraconsistent logic does not exist. His claim is based on the fact that, according to him, paraconsistent negations are not negations. The reason why is not really convincing.[1]

But even if until now no one has really disproved that paraconsistent negations are negations, no one has provided serious philosophical or matemathical reasonings and evidences to show that they are. So the question of the existence of paraconsistent logic is still an open problem.

In a first part we deal with a general philosophical discussion about the possible existence of paraconsistent logic. The second part will be a more technical discussion about the properties of "existing" paraconsistent negations.

1 Existence, irrationality and confusion

1.1 The relative existence of paraconsistent logic: paraconsistentology, paraconsistentists and paraconsistent logic

In some sense paraconsistent logic exists : many different systems of paraconsistent logics have been presented and studied over the years. A section of *Mathematical Reviews* has been created. Three world congresses have been organized (one in Belgium, one in Poland and the present one in Brazil). But one must make a clear distinction between the subject and the object. What exists in fact are people, let us call them, *paraconsistentists*, who study some supposed paraconsistent logics. Their science can be called *paraconsistentology*, or paraconsistency for short (although we will not use this term since in general people confuse paraconsistency with paraconsistent logic and the point is to show the difference). For the present time there are no clear evidences of the existence of paraconsistent logic, but paraconsistentists and paraconsistentology exist for sure, and even if one day someone proves that paraconsistent logic does not exist, this will not necessarily entail the non existence of paraconsistentists and paraconsistentology.

Let us explain this by a metaphor. Imagine that there is a planet in the universe called Babakos whose inhabitants are called Babakons, and let us call the specialists of Babakons, Babakonologists and their science Babakonology. Imagine furthermore that the existence of the inhabitants of Babakos is not certain. The

[1] See [2], [3] and [4] for a criticism of Slater's arguments.

planet is very far from the Earth and the observations about the planet are not sufficient to guarantee their existence, although there is some kind of evidence of their existence. Imagine that one day it is proven that there are no Babakons. So what about Babakonologists and Babakonology? Certainly the proof, by observations and/or theoretical means, of the nonexistence of Babakons can be considered as part of Babakonology. And this is certainly not the end of Babakonologists, since they probably have a lot to say about the nonexistence of Babakons and this knowledge can be useful for the study of the existence or nonexistence of inhabitants of other planets in the universe. Babakonology is part of the study of ET-life (extra-terrestrial life) and it has a value as such whether Babakons exist or not.

Let us emphasize that (before any proof of the existence or nonexistence of Babakons is given) the *belief* in Babakons is independent of Babakonology. Someone who believes in Babakons is not necessarily a Babakonologist and someone who doesn't believe in Babakons can be a very good Babakonologist. The belief in Babakons can be a good motivation for one to turn into Babakonology, but the disbelief can also be a strong impulse.

Someone may believe that Babakons do not exist and for that reason be against the development of Babakonology. For example one may, for religious reasons, think that humans are the only beings in the universe. This kind of behaviour is not good for Babakonology, nor for science in general. A no better position would be the situation of someone who believes that there are people living in the Moon. His belief may be based on a book from, let us say, Ancient Egypt and he will try by any means, fractal topology, quantum astronomy, bi-polar logic, to prove the existence of inhabitants in the Moon.

The present situation in mathematical logic does not prove or disprove the existence of paraconsistent negations and we must keep in mind two things: 1) We cannot infer the existence of God from the existence of theology; 2) We cannot infer the nonexistence of God from the existence of atheists. This means: we cannot infer the existence of paraconsistent logic from paraconsistentology, or from the existence of paraconsistentists; we cannot deny the existence of paraconsistent logic just because there are people who don't believe in it.

1.2 Comparison with the existence of other logics

The situation of paraconsistent logic is quite different from the situation of some other logics. Let us take the example of the fashionable *linear logic*. At the present time there are no serious doubts about the existence of such a logic, nobody has written a paper entitled "Linear logic?", trying to show that there are no linear logics. The reason is very simple and can be found in the very name "linear logic". It is a technical mathematical term without any philosophical connotations. This term is related to some mathematical background which was, according to Girard, the origin of the idea of linear logic. However there are no clear connections between the mathematical background of this term and the philosophical ambitions of linear logic.

Slater deduces that there are no paraconsistent logics from an alleged proof that a negation which is paraconsistent is not a negation. In linear logic all the connectives are different from the classical ones. Someone maybe can say that linear negation is not a negation and that therefore there are no linear logics. But the aim of linear logic is not to provide a new negation, it is more general: to provide logical operators which are adequate to deal not with eternal truths but perishable recyclable data. It is very difficult to know exactly to what extent linear logic is a satisfactory solution to the problem. There is certainly a huge gap between the vernacular examples presented by Girard to motivate his logic and the way linear logic works. At the end the question is: "Do the mathematical operators developed match some operations of any sort of reasoning?". If it is not the case, one can claim, not that linear operators don't exist, but that linear logic doesn't exist, simply because it is not a *logic*. What exists is a mathematical system, and many mathematical systems have nothing to do with logic. It would be the same situation as if Babakonologists discovered that a kind of monkeys were living on Babakos instead of something similar to humans.

In some sense linear logic is the result of a formal game which consists of modifying a mathematical tool, sequent calculus, developed to represent classical logic. This can make sense if later on an interpretation is provided. This is the main problem not only of linear logic but of the other *substructural logics* which are obtained by modifying the structural rules of sequent calculus.

A similar situation is that of *many-valued logic*, which is the result of generalizing the standard two-valued matrix of classical logic, considering matrices with more than two values. Here we have the same question as in the case of substructural logic: to know if the operators defined by many-valued matrices have a logical interpretation, that is to say the question to know if many-valued logic is really a *logic*. As it is known, Łukasiewicz developed many-valued logic in order to catch the notion of possibility, but it seems that with many-valued logic it is not possible to properly define this notion. Even if one really succeeds to give a meaning to operators of many-valued logic, there is still the question of whether many-valued logic is really *many-valued*. Suszko has shown that it was possible to provide a two-valued semantics for Łukasiewicz's three-valued logic and Suszko pointed out that we must not confuse logical values with algebraic values (on this topic see [5], [6], [7]).

In the case of *modal logic*, we have mathematical operators designed to represent the notions of possibility and necessity. In fact there is a whole class of modal logics and it is not clear at all which modal logic represents rightly these notions, if any. In this sense the question of the existence of *modal logic* is still an open one. Modal logic has been developed these last years in a pure mathematical way as the general study of unary operators. One asks for example which kind of unary operators can be represented by a Kripke structure. In this sense modal logic includes paraconsistent logic, since negations are unary operators. Nobody seems to be aware of this fact and this shows very well that most of the people working in modal logic do not really think about the interpretation of the "modal" operators.[2] It is not necessary wrong to call these operators "modalities", since a modality in fact is any variation of a given statement, including negation and affirmation. From this point of view, if the existence of paraconsistent logic was proved this will entail the existence of

[2]Of course many interpretations are in the air: knowledge, belief, information, etc. But one thing it to have a general intuitive idea and another one is to carry on a systematic investigation to see if the mathematical properties really fit with the interpretation. The difficulty here is that on the one hand we have something precise and the other hand something rather fuzzy, one has to check if a precise thing match with a fuzzy thing.

modal logic, at least a modal logic different from classical logic, since classical logic can itself be considered as a modal logic.

1.3 The supremacy of classical negation

Classical negation of mathematical logic (hereafter *Clanemalo*) is one representation of negation. The claim that it is the right representation of negation is very controversial. A less controversial claim is that classical negation is the right representation of negation as it appears in mathematical reasoning. An even less controversial claim is that it is the right representation of negation as it appears in *classical* mathematical reasoning.

Outside the sphere of mathematical reasoning, negation appears in many forms, some of them having very few connections with Clanemalo, so for this reason it seems totally absurd to say that Clanemalo is the right and only negation. Mathematical reasoning is certainly different from vernacular reasoning.

One may think however that mathematical reasoning is the only right reasoning, that vernacular reasoning is obscure and ambiguous and that Clanemalo is the only right negation, that vernacular negation is obscure and ambiguous. In this case Clanemalo has to be clearly taken as a *normative* definition of negation and not a *descriptive* one, since it does not describe properly vernacular negation.

One may want to give a descriptive definition, through mathematical logic, of vernacular negation. But this is not necessarily an obvious task. The classicist says: "Vernacular negation is ambiguous, Clanemalo works good, we must use Clanemalo". This leaves open the question of why, how and to which extent vernacular negation is ambiguous. The anti-classicist thinks that Clanemalo is a caricature, that it is not a good picture of *real* negation. He thinks that the classicist pejoratively says that vernacular negation is ambiguous only because it does not fit into the simplified schema of Clanemalo. He would say that vernacular negation is not ambiguous, but more *complex* that the oversimplified Clanemalo. Trying to give some other representations of negation the anti-classicist may shed som light on the nature of vernacular negation.

On the one hand the classicist tends to reject vernacular negation as ambiguous preferring a pure platonic idealization, on the other hand the anti-classicist tends to venerate the vernacular negation, with a kind of blind respect for concrete empirical data.

Maybe it is important to recall to someone fascinated by the "incredible complexity" of vernacular negation and who doesn't want to deal with classical negation, that one of the fundamental basis of science is the process of abstraction by simplification. It is interesting here to recall what Gentzen was saying about constructivist mathematics versus classical mathematics:

> We might consider still another example which, in its relation to physics, seems to provide even more striking analogies to the relationship between constructivist mathematics and actualist mathematics:
>
> I am thinking of the occasional attempt to construct a 'natural geometry', i.e. a geometry which is better suited to physical experience than the usual (Euclidean) geometry, for example. In this natural geometry, the theorem 'precisely one straight line passes through two distinct points' holds only if the points are not lying too close together. For if

> they are lying very close together, then *several* adjacent straight lines can obviously be drawn through the two points. The *draftsman* must take these considerations into account; in *pure geometry, however*, this is not done because here two points are *idealized*. The *extended* points of experience are replaced by the *ideal*, unextended, 'points' of theoretical mathematics which, in reality, have no existence. That this procedure is beneficial is borne out by its success: It results in a mathematical theory which is of a much simpler and considerably smoother form than that of natural geometry, which is continually concerned with unpleasant exceptions.
>
> The relationship between actualist mathematics and constructivist mathematics is quite analogous: Actualist mathematics idealizes, for example, the notion of 'existence' by saying: A number *exists* if its existence can be proved by means of a proof in which the logical deductions are applied to *completed infinite* totalities in the same form in which they are valid for finite totalities; entirely as if these infinite totalities were actually present quantities. In this way the concept of existence therefore inherits the advantages and the disadvantages of an ideal element: The *advantages* is, above all, that a considerable simplification and elegance of the theory is achieved - since intuitionist existence proofs are, as mentioned, mostly very complicated and plagued by unpleasant exceptions-, the *disadvantage*, however, is that this ideal concept of existence is no longer applicable to the same degree to physical reality as, for example, the constructive concept of existence. (...)
>
> The question now arises: what use are elegant bodies of knowledge and particularly *simple* theorems if they are not applicable to physical reality in their literal sense? Would it no be preferable in that case to adopt a procedure which is more *laborious* and which yields *more complicated* results, but which has the advantage of making these results immediately meaningful in reality? The answer lies in the *success* of the former procedure: Again consider the example of geometry. The great achievements of mathematics in the advancement of physical knowledge stem precisely from this method of *idealizing* what is physically given and thereby *simplifying* its investigation. ([8], pp.248-249).

It is important to keep Gentzen's remarks in mind at a time where a lot of intricate ugly "draftsman logics" are presented, which contrast so much with the beauty and simplicity of classical logic. Of course one can think that to venerate only classical logic and Clanemalo would be the same as thinking that natural numbers are very nice and that we don't need real numbers, ugly ambiguous imprecise things. But the people who think that Clanemalo is an absurd simplification of vernacular negation that must be banished certainly are not aware of the incredible jump that was made in Greece, more than two thousands years ago, when the people started to use the principle of non-contradiction (hereafter PNC). It is probably not wrong to say that the use of the principle of non-contradiction was the start of mathematics and science in general. Interesting enough the appearance of the PNC coincides with a rejection of empiricism.[3]

[3] On this topic see the remarkable book of Szabo [9]. What Szabó discusses is essentially the

One can have two opposite perspectives on paraconsistent negation:

1. Paraconsistent negation would be something less idealized than classical negation, and paraconsistent logic would be like a draftsman geometry.

2. Paraconsistent negation can be seen as an extension of the sphere of rationality, in the same sense that irrational numbers or transfinite numbers can be conceived as an extension of the sphere of rationality rather than a drawback toward a "draftsman mathematics".

It seems that the terminology "Transconsistent Logic" coined by G.Priest [10] is good to express this second perspective.[4] This second perspective, which can also be traced back to Vasiliev with his notion of "Imaginary Logic" or "Non-Aristotelean Logic" (on Vasiliev see for example [11], [12]), is very challenging but also is very controversial for several reasons. If we have a look at the birth of science in the Greek world, the PNC can itself be considered as the foundation of rationality, so it is not quite the same to consider the move from natural numbers toward irrational numbers and the move from classical negation toward paraconsistent negation. Of course the "crisis" of irrationals was really a crisis, but it was not a crisis of rationality, despite the expression "irrational".

People sometimes like to make an opposition between occidental rationality based on the PNC and oriental wisdom. This is the case of Kosko in a popular book about fuzzy logic (cf [13]). Anyway we must recall that within the occidental Greek tradition there were people like Heraclitus or Hegel who defended a kind of rationality not based on the PNC, even based in fact on something which appears as the contrary of the PNC. But if we look at the history of science, we see that until now this has led to nowhere. Maybe paraconsistentology can be the first stone in the construction of a new rationality, but we still don't know if any building based on paraconsistent logic will stay erect.

1.4 Irrationality and paraconsistency

Some people think that paraconsistent logic is dangerous, that to give away the PNC will lead to nonsense, chaos, confusion. They think that the PNC is the foundation of rationality, and that without it, there will be no more distinction between truth and falsity. Similar criticisms have been addressed to fuzzy logic.

Of course such kind of criticisms make sense in a world where we are surrounded by contradictory statements by politicians, advertisements and in fact at all levels of information. Someone who is supporting paraconsistent logic may appear as supporting the surrounding confusion.

Can we take seriously someone who says "I believe in God and I do not believe in God"? The classical rationalist will say no. But someone can say: "Well, not everything in life is black or white, it can be grey (cf. the famous "grey zone" of Kosko [13]). Someone can be beautiful and ugly, republican and Quacker, rich and communist". We must be very careful at this point because there are several important issues which are mixed.

use of the *reductio ad absurdum*, which is the strongest form of the PNC, in particular Clanemalo can be defined only with the *reductio*.

[4]The philosophical position of Priest himself is however not very clear, sometimes it seems that his perspective rather falls under 1.

If someone asks us "Do you believe in God ?", we can have no answer to this question, we can say "This question makes no sense to us because we don't know exactly what do you mean by God", or we can say "In some sense we believe in God, in some other sense we don't believe in God". Does this mean that we are rejecting the PNC? Not necessarily.

Given a property P, the PNC divides a class of objects into two parts, the objects having this property, and the objects not having this property. Let us say that the property is "to be odd", using the PNC, we have the class of odd numbers and of non-odd numbers. Now we can ask: "Is God odd?". The question makes no sense because odd is a property which applies to numbers only. Someone could say "God is both odd and even", and claims that the PNC is not valid. But in the best case, this has to be taken only as (bad) poetry and not a serious challenge to the PNC.

A number which is not odd is called even. Even means nothing more, nothing less that non-odd. It is clear that in natural language there are a lot of pairs of words that don't work like that, for example the pair blonde/intelligent: a woman can be blonde and intelligent without infringing the PNC. This at first seems a kind of terrible triviality. But it seems that this triviality is not so blatant for some of the people who want to reject the PNC.

However what is very interesting in paraconsistentology is the attempt to develop a negation which should be able to deal with pairs of concepts which work in a way very similar to contradictories but at the same time admit a common intersection.

On the other hand there are no good reasons to radically reject the PNC. It is clear that the PNC is useful in some sense, and that it is working quite well in many situations. Paraconsistent logic is not in fact necessarily based on a rejection of the PNC. If we define, as we did, paraconsistent logic as a logic in which there is a paraconsistent negation, then we may also have a classical negation, therefore in this case a paraconsistent logic is an extension of classical logic. A paraconsistent negation is an additional operator. Sometimes, like in the case of da Costa's logic $C1$, it is possible to define the classical negation with the paraconsistent negation. And what about the converse? We have pointed out that it is possible to define something which looks very much like a paraconsistent negation within first-order logic (cf [14]).

From this point of view it is clear that paraconsistent logic appears rather as an extension, than a rejection of classical rationality. If it exists ! Because we must not play with words, we still don't really know if there are any paraconsistent logic.[5]

1.5 Three ways to confusion

According to Slater (see [1]), the existence of paraconsistent logic is a result of a verbal confusion. Paraconsistent logic are dealing with subcontraries and not contradictories. Slater claims that paraconsistentists say that they are talking about negation, because they switch contradictories for subcontraries. It would be the same as to call women "men" and men "women". In this case one would be able to claim that women produce sperm, but of course the reality would not have

[5] About the topic of paraconsistency and irrationality one may consult the interesting book of G.G.Granger [15] dedicated entirely to irrationality and which inlcudes a chapter on paraconsistent logic.

changed. Let us call this kind of confusion *switching confusion.* If someone claims that women produce sperm, it can be the result of an important discovery about the physionomy of women or just the result of a switching confusion. Two very different cases indeed.

We have shown in another paper (cf [4]) that paraconsistentists cannot be accused of such an easy trick,[6] that paraconsistent logic is not the result of a switching confusion, but it may be the result of other confusions.

Imagine that we extend the concept of inhabitants in order to include monkeys, dogs, or even rats and that small rats are discovered on Babakos, then we can say that there are inhabitants on Babakos. In the same way, if we extend the notion of negation to any unary operator, then we can say that paraconsistent negations exist. We can call this kind of confusion *global confusion.* Certainly many paraconsistentists make implicitly this kind of global confusion when they start to speak about such or such operator they called paraconsistent negation just because it does not obey the PNC.

Finally we would like to talk about *Christopher Columbus* confusion. As it is known Columbus wanted to go to India, but he reached America instead and the inhabitants of America were called "Indians" because at first he thought that he was in India. The discover of America was a very important fact, but of course "Indians" are not Indians. This does not mean that they don't exist or that they are not interesting people, but they are different kind of people. One can say in some sense that Łukasiewicz made a kind of Columbus confusion. He wanted to reach modalities, but reached something else by many-valued logic. Maybe it is what is happening with paraconsistentists. They are looking for negations, but perhaps the operators they are discovering are something else, very interesting, but not negations. And perhaps, in the same way that nobody calls nowadays Łukasiewicz's logic $L3$ a modal logic, nobody will call in the future $C1$, LP or $P1$ paraconsistent logics. It is true that sometimes the power of words is really strong and that until now it is still quite common to call "Indians" people originally from North or South-America, although the terminology "American Indians" has been introduced. Anyway the important thing is that despite the name, few people believe that these Indians are from India.

Paraconsistentists may escape the most trivial confusion, the switching confusion, but it seems that in general they are not very careful about the use of the word "negation", they eat their cake before cooking it and made a lot of global confusions, and at the end they may even be into a big Colombus confusion.

[6] As Slater rightly recalled during our talk at the WCP2, he just generalized an idea which was first proposed by R.Routley and G.Priest in [16]. Routley and Priest were arguing that da Costa's negation was not a negation, that it was a subcontrary forming relation rather than a contradictory one. Slater showed that their argument could also be applied to Priest's negation and any paraconsistent negation. Later on Priest recognized that we should rather considered erroneous his original argument against da Costa's negation than to think that Slater's generalized argument is right.

2 A guided tour in the land of paraconsistency

The question to know if paraconsistent negations are really negations, if there really is any paraconsistent logic, must necessarily lead to a systematic study of the technical aspects of paraconsistent negations.

All paraconsistentists are united by a negative criterium: the rejection of the *ex contradictione sequitur quod libet* (EC for short). Mathematically speaking they say that if a negation $\neg$ is *paraconsistent* then

$$a, \neg a \nvdash b$$

Note that it would be absurd to say that if $a, \neg a \nvdash b$ then $\neg$ is paraconsistent. It is clear that the rejection of EC is a necessary condition but not a sufficient one. In order to be a paraconsistent *negation*, $\neg$ must have some *positive* properties.

On the one hand it is not clear at all which properties are enough to define a negation. On the other hand given a set of properties for negation, one has to investigate if these properties are compatible. So let us say that people agree that a given set of properties SCN is enough to define a negation, then one has to check if this set together with the rejection of EC form a compatible set of assumptions. One may want to prove that there are no paraconsistent negations by considering a set NCN of necessary properties for negation and showing that it is not compatible with the rejection of EC.

The point is that it is not clear what should be sets of properties like SCN and NCN. The question is difficult for mainly two reasons: on the mathematical side, the propreties for negation can be of very different natures, on the philosophical side, it is not easy to have a coherent and intuitive interpretation of an operator having such or such property.

To study this problem one has first to describe and classify the properties of negation. This is what we will do in the next section. Then there are two methods: you can construct a logic, showing that some given properties are compatible, or you can get negative results showing that given properties are not compatible. What did happen in the field of paraconsistency until now is closer to the first method: people have built logics. But most of the time they claim that these logics are paraconsistent without investigating really which properties the underlying "negations" have and if these properties are sufficient to justify the name.[7]

In the next sections we will make critical reviews of the main paraconsistent negations which have been presented thus far.[8]

[7]During many years negation was not a notion in focus. Different negations were presented, like intuitionistic or minimal negations, but negation was not by itslef a subject of a systematic investigation. An exception maybe is the work of Curry. In [17] he presented a comparative study of four types of negation. This work is very interesting but Curry's treatement does not allow paraconsistent negations (in a previous paper [18], we have presented a generalization of Curry's work). A couple years ago, Gabbay started to investigate negation [19], trying to propose a definition and so on. Following his interesting initiative, a group of people have work on this direction (cf. [20], [21]). However until few years ago they didn't know nearly nothing about the extended literature on paraconsistent logic (it is obvious when we look at the lists of references of their works), and didn't include explicitly paraconsistent negation in their treatement.

[8]Not every paraconsistent negation "existing" under the sun will be discussed. Despite the present high speed of circulation via information highways, we are not necessary aware of everything, of some exotic paraconsistent negations elaborated in the florest of Transylvania, where the access to the internet is still limited.

2.1 Classification of the properties of negation

The classification of properties for negation depends on a general framework for the study of logics. Following our idea of *Universal Logic* (cf [22], [23], [24]), we just consider a *logic* as a mathematical structure of type

$$\langle L; \vdash \rangle$$

where L is any set and $\vdash$ is a relation between sets of objects of L and objects of L. A *negation* is a function defined on L having such or such property.[9]

We will use the abbreviation $a \dashv\vdash b$ for $a \vdash b$ and $b \vdash a$.

Pure laws[10]

1. Reductio ad absurdum[11]
if $\neg a \vdash b$ and $\neg a \vdash \neg b$, then $\vdash a$ [12]
if $a \vdash b$ and $a \vdash \neg b$, then $\vdash \neg a$
if $\neg a \vdash a$ then $\vdash a$
if $a \vdash \neg a$ then $\vdash \neg a$

2. Contraposition
if $\neg a \vdash \neg b$ then $b \vdash a$
if $a \vdash b$ then $\neg b \vdash \neg a$
if $a \vdash \neg b$ then $b \vdash \neg a$
if $\neg a \vdash b$ then $\neg b \vdash a$

3. Double negation
$\neg\neg a \vdash a$ [13]
$a \vdash \neg\neg a$

The two fundamental laws

1. Law of non-contradiction (LNC for short)
$\vdash \neg(a \wedge \neg a)$ [14]
2. Law of excluded middle (EM for short)
$\vdash a \vee \neg a$

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[9] As the reader should have understood after our discussion in the first part of this paper, not every function defined on L can be called a "negation".

[10] By *law* we mean here statements about the relation $\vdash$. This relation is not considered as a proof-theoretical notion. It is important not to confuse a law with a *rule of deduction*, mistake too much common in the literature nowadays. The properties of negations that we present here are not proof-theoretical properties, even less syntactic properties. They are properties of a function in a structure.

[11] The different forms of *reductio ad absurdum* are not equivalent, but there are good reasons to use the same generic name for these four versions; on this topic the reader can consult the book of J.-L.Gardies [25] entirely dedicated to the *reductio*.

[12] This should be considered as an abbreviation of the following statement: for any set of formulas T, and any formulas a and b:
if $T, \neg a \vdash b$ and $T, \neg a \vdash \neg b$, then $T \vdash a$
This kind of abbreviation can be ambiguous since in substructural logics, including relevant logic, this abbreviation is not always equivalent to the thing abbreviated. Anyway we will use it and also use the same kind of abbreviations hereafter for the other laws.

[13] This law could be expressed equivalently, modulo some basic properties of $\vdash$, in the following way:
if $\vdash \neg\neg a$ then $\vdash a$.
For more details about this, see [18].

[14] This law of non-contradiction LNC should not be confused with the informal principle of non-contradiction PNC, one may think that the correct formulation of PNC is the *ex contradictione* EC ; for more discussion about this, see the section about full paraconsistent logic.

De Morgan Laws

1. De Morgan laws for conjunction

$\neg(a \wedge b) \dashv\vdash \neg a \vee \neg b$

$\neg(\neg a \wedge \neg b) \dashv\vdash a \vee b$

$\neg(\neg a \wedge b) \dashv\vdash a \vee \neg b$

$\neg(a \wedge \neg b) \dashv\vdash \neg a \vee b$

2. De Morgan laws for disjunction

$\neg(a \vee b) \dashv\vdash \neg a \wedge \neg b$

$\neg(\neg a \vee \neg b) \dashv\vdash a \wedge b$

$\neg(\neg a \vee b) \dashv\vdash a \wedge \neg b$

$\neg(a \vee \neg b) \dashv\vdash \neg a \wedge b$

3. De Morgan laws for implication

$a \rightarrow b \dashv\vdash \neg a \vee b$

$\neg a \rightarrow \neg b \dashv\vdash a \vee \neg b$

$\neg a \rightarrow b \dashv\vdash a \vee b$

$a \rightarrow \neg b \dashv\vdash \neg a \vee \neg b$

Self-Extensionality

The property of self-extensionality, expression coined by Wójcicki (see [26]; and [27] for comments about this terminology), corresponds to the validity of the replacement theorem and can be expressed in the following way:

if $a \dashv\vdash b$ then $T \vdash c$ iff $(T \vdash c)[b/a]$

where $(T \vdash c)[b/a]$ means that b replaces a in T and c.

Representability properties

Several general properties related to the *representability* of logics can be considered: a negation is *truth-functional* iff it can be expressed by a finite matrix, it is *leibnizian* iff it can be described by a possible world semantics, it is *effective* iff it can be defined with a recursive proof-system, etc...

Other properties

Now let us finish by stating properties not directly connected with negation but which are important for the discussion. A paraconsistent logic has generally the same basic language as classical logic, that is to say we have a negation, and three binary connectives: a conjunction, a disjunction, and an implication. Of course since the negation of a paraconsistent logic has not the same features as the negation of classical logic, the binary connectives cannot have exactly the same behaviour as the classical one, if we take into account the fact that the behaviour of a connective depends on the whole context. Anyway the binary connectives can be quite similar to the classical ones in the sense that a paraconsistent logic can be a conservative extenstion of *positive classical logic*. On the other hand we can have a paraconsistent logic where this does not happen. There are mainly two basic cases:

1. *Non adjunctive paraconsistent logics* These are logics in which the conjunction fails to obey the following law of adjunction: $a, b \vdash a \wedge b$

2. *Non implicative paraconsistent logics* These are logics in which the implication fails to obey the following law of implicativity: if $\vdash a \rightarrow b$ then $a \vdash b$.

We have summarized above the main properties a negation can have.[15] Obviously the strongest properties are the various laws of *reductio ad absurdum* and contraposition. Unfortunately none of these, except the last two forms of *reductio*, are compatible with the rejection of EC. Or to be more exact with the rejection of EC and its weak form: $a, \neg a \vdash \neg b$. For a detailed account about this fact see [18].

So the only hope for the paraconsistentist is to gather other properties.

2.2 Non self-extensional paraconsistent logics

Most of the well-known non classical logics, like modal logics, intuitionistic logic, linear logic, etc., are self-extensional. Many people think that a logic must be self-extensional. However this is rather because it is a nice technical and practical property than for any precise philosophical reason. As we have argued elsewhere (cf [29]), there are no reasons *a priori* to reject a logic just because it is not self-extensional. Moreover it seems that any logic that wants to capture intensionality should be non self-extensional (cf [27]).

Nevertheless if a logic is not self-extensional, the counter examples of self extensionality must have an intuitive explanation. Unfortunately it seems that it is not the case with several paraconsistent logics which are not self-extensional.

In da Costa's logic $C1$, the formulas $a \wedge b$ and $b \wedge a$ are logically equivalent (i.e. $a \wedge b \dashv\vdash b \wedge a$) but not their negations and nobody has presented a philosophical idea to support this failure. In Priest's logic LP and in da Costa and D'Ottaviano's logic $J3$, the formulas $a \vee \neg a$ and $b \vee \neg b$ are logically equivalent but not their negations and here again no philosophical justification for this failure has been presented.[16]

Several results show that some properties of negation are incompatible with the idea of a self-extensional paraconsistent negation (see [35]). Maybe one can conclude from this that paraconsistent negations are not negations. However this will we a controversial conclusion as long as one gives a convincing reason why a negation should be self-extensional. On the other hand one may argue that it is not a problem since a paraconsistent negation should be an intensional operator. First, let us note that not any non self-extensional operator is intensional. Second, one should be able to provide an intuitive explanation of the failure of the replacement theorem, based on a discussion about intensionality or not.

2.3 Full paraconsistent logics

We say that a paraconsistent logic is full when we have:

$\vdash \neg(a \wedge \neg a)$.

There was a time when the people didn't make any distinction between this law of non-contradiction LNC and EC. In fact the question is still open to know if we can find an intuitive interpretation of an operator which obeys EC and not LNC or obeys LNC and not EC.

In the three-valued logic $L3$ of Łukasiewicz, LNC is not valid, but EC is. And this seems quite odd following the interpretation of his third value. If the value of a formula and its negation are undetermined, then the value of their conjunction

[15] Notice that we didn't mention non classical properties, that is to say properties which are not valid for classical negation, like for example: if $\neg a \vdash a$. For a discussion about this, see [28].

[16] about $C1$, see [30] and [31]; about LP, see [32] and [10]; about $J3$, see [33] and [34]

is undetermined and so is the negation of this conjunction. As undetermined is not a distinguished value, then LNC fails; but at the same time why should one be able to deduce anything from a formula and its negation when they are both undetermined?

There are many three-valued paraconsistent logics where the negation is defined exactly in the same way as in Łukasiewicz's logic, but the undetermined value is considered as distinguished, the effect of this interchanging is that LNC is valid but not EC. The problem is that the same oddity as in $L3$ appears in an inverted way.

The gap between LNC and EC is in fact not so huge. How can one jump from LNC to EC? This can be done in three easy steps: by the use of self-extensionality, involution[17] and adjunction (see [35]).

At the end it is not clear at all that the idea of a full paraconsistent logic is meaningful. In fact the initial idea of da Costa, the Pope of paraconsistent logic, was to reject both LNC and EC.

2.4 Paraconsistent classical logics

Paraconsistent classical logics (hereafter PCL) are logics which have the same theorems as classical logic, they differ only at the level of the consequence relation.

Note therefore that any PCL is full, so that a paraconsistent negation in a PCL cannot be at the same time involutive, self-extensional and adjunctive (due to results of [35]). Priest's logic LP is a PCL which is involutive but not self-extensional. Urbas's dual-intuitionistic logic LDJ is a PCL which is self-extensional but not involutive. Jaśkowski's discussive logic is a PCL which is self-extensional, involutive but not adjunctive.[18]

Moreover a PCL cannot be implicative, since in a PCL we have

$\vdash a \rightarrow (\neg a \rightarrow b)$.

If it is implicative we will get

$a, \neg a \vdash b$.

So Priest's logic LP, Urbas's LDJ and Jaśkowski's discussive logic are all non implicative.

What is the problem with non implicative logics? In a logic which is not implicative, there are formulas a and b such that

$\vdash a \rightarrow b$ and $a \nvdash b$

therefore under the assumption of the transitivity of $\vdash$ in such a logic the following version of the *modus ponens* cannot be valid

$a, a \rightarrow b \vdash b$.

Under the assumption of monotonicity the following version of the *modus ponens* cannot be valid

if $\vdash a$ and $\vdash a \rightarrow b$, then $\vdash b$.

The aim of this paper is not to discuss implication, but one can think that an implication without *modus ponens* is something as paradoxical as free money.[19]

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[17] A negation is said to be involutive when both double negation laws hold.

[18] On Urbas's logic, see [36]; on Jaśkowski's logic, see [37] and [38]; a general paper on PCL is [39].

[19] There are a lot of discussions about implication, and many people, like relevantists, think that a connective which does not obey the *modus ponens* can still be called an implication. We have here a problem similar with paraconsistent negation. Relevantists are probably not making a switching confusion, but maybe a global confusion or a Columbus confusion (cf Section 1.5) Some

2.5 Paraconsistent atomical logics

Paraconsistent atomical logics (PAL for short) are logics where only atomic formulas have a paraconsistent behaviour. Molecular formulas have a standard behaviour. That means you can have

$a, \neg a \nvdash b$

only when a is an atomic formula.

The intuitive motivation can be the following: contradictions may appear at the level of facts, data, information (or whatever the atomic formulas are supposed to represent) and at this level they should not entail triviality, but reasoning is classical, so when you go to the logical level, i.e. non atomical, everything should work in the normal classical way.

This idea at first seems reasonable, but if you think about it two seconds you will see that PAL sound quite absurd: for example from an atomic formula a and its negation $\neg a$ you cannot deduce anything but you can do so from the molecular formula $a \wedge a$ and its negation $\neg(a \wedge a)$. One can seriously wonder how such duplication can draw a line between the field of paraconsistent reasoning and the field of classical reasoning. Moreover this same example shows that any PAL is not self-extensional.

So PAL seem really not a good solution to the paraconsistent problem, i.e. the problem of finding a negation which is paraconsistent and has a coherent intuitive interpretation. Several PAL have been presented. Sette's logic $P1$ is one of them [40]. It is interesting to remember how this logic was created. Sette wanted to solve the maximal problem set by da Costa and he presented $P1$ as a solution to this problem. $P1$ is certainly a solution to this problem, but the example of $P1$ just shows that solving this problem does not necessarily solve the paraconsistent problem.[20] If we are interested in paraconsistent logic, we must always keep the paraconsistent problem in mind which involves both technical and philosophical aspects. Blind technicality can be a fun game for those who practice it but most of the time leads only to formal nonsense.

Another PAL is the logic V, which has been presented by Puga and da Costa as a possible formalization of Vasiliev's logic (cf [41]). From the above considerations, we can conclude that either V is not a good formalization of Vasiliev's logic, either Vasiliev's logic is not a good solution to the paraconsistent problem.[21]

2.6 Truth-functional paraconsistent logics

In the subsection about full paraconsistent logics, we already said a word about three-valued logics. We noticed that some of them have the same negation as the one of Łukasiewicz's logic $L3$. The only difference is that the undetermined value in these paraconsistent logics is considered as distinguished. We have seen that this leads to a strange unsatisfactory feature.

people think that relevant implication is the right implication and that classical implication should not be called implication. But implication of classical logic represents perfectly the implication of mathematical reasoning, which is certainly an important aspect of human reasoning.

[20] The maximal problem is the problem of finding a paraconsistent logic which is maximal in the sense that it has no strict extension other than classical logic or the trivial logic; J.Marcos [34] has recently shown that there exists more than 8K solutions to this problem, and most probably some of these solutions are more interesting from the viewpoint of paraconsistency than Sette's $P1$.

[21] For more discussion about this question see [42]

What are the other possibilities? There are not a lot of them: the other ways to define another kind of paraconsistent negation in a three-valued logic leads, either to the validation of LNC (so we get the same problem as before), or to a paraconsistent atomical logic, as the reader can easily check.[22]

Maybe the only solution to get a reasonable paraconsistent negation it to work with four values. This has not yet been investigated systematically. The only four-valued paraconsistent logics which have been really studied are Nelson's logic (cf [43], [44]) and Belnap's logic. In Belnap's logic neither LNC, nor EC are valid, but the excluded middle is not valid either and if you add it to Belnap's logic you get classical logic.[23]

2.7 Paraconsistent morganian logics

We say that a paraconsistent logic is morganian if all De Morgan laws are valid as well as the two double negation laws. In most paraconsistent logics you have only some parts of the De Morgan laws. There are also paraconsistent logics in which all De Morgan laws for conjunction and disjunction are valid. This in particular the case of standard truth-functional paraconsistent logics like J3, LP or Belnap's logic (see also [49]).

Paraconsistent morganian logics cannot be self-extensional (on the assumption of adjunctivity). Let us prove this:

From the fact that we obviously have $\vdash a \rightarrow a$, we get $\vdash \neg a \vee a$, by application of the De Morgan law for implication. So we have $a \vee \neg a \dashv\vdash b \vee \neg b$. By self-extensionality, we get $\neg(a \vee \neg a) \dashv\vdash \neg(b \vee \neg b)$. Now applying De Morgans's law for disjunction and self-extensionality (or transitivity), we get: $\neg a \wedge a \dashv\vdash \neg b \wedge b$. From this it is easy to see that we have $\neg a \wedge a \vdash b$. And finally applying adjunction, we get $\neg a, a \vdash b$.

From this proof it is possible to see also that a logic cannot admit the excluded middle, be morganian, self-extensional and paraconsistent.

This fact corresponds to an obvious algebraic result: if you have a De Morgan lattice and you add the excluded middle, it will be the greatest element of the lattice, now by De Morgan law, this means that there is also a smallest element, which is of the form $a \wedge \neg a$ (see [50]).

In conclusion: the problem with morganian paraconsistent logics is that we have to choose between self-extensionality and the excluded middle.

[22] This is true unless one admits a non conservative matricial definition of conjunction (i.e. conjunction defined as *min*) or is ready to consider as negation, an operator such that $a \leftrightarrow \neg\neg\neg a$.

[23] The reader can find a good study of three-valued paraconsistent logics in [34], and [45]. [46] is a tentative to generalize truth-functional semantics which could be fruitful for the developement of paraconsistent logic. About Belnap's logic, see [47]. Belnap didn't know what was paraconsistent logic when he developed his logic. The fact that Belnap's logic is paraconsistent was already noticed by da Costa in his *Mathematical Review* (58 5021) of Belnap's paper. But of course it is not clear to which extent Belnap's negation is a paraconsistent *negation*.

Historically it seems that the first person who had the idea to use logical matrices to develop paraconsistent logic was Asenjo (see [48]). He was followed later on by da Costa and D'Ottaviano with their system $J3$, Sette with his system $P1$ and finally by Priest with his system LP.

2.8 Paraconsistent leibnizian logics

By paraconsistent leibnizian logics we mean logics constructed with a possible world semantics.

There are two very simple ideas that we will discuss here. They are not of course the only possible but it seems that they are the two basic ones. Furthermore they are the only two which have been investigated in details, so in the present paper we will limit our discussion to them.[24]

We recall that possible world semantics is based on the idea to consider sets of possible worlds. In the basic semantics discussed here, no relation of accessibility is involved (the same as to consider a universal relation of accessibility). Possible worlds can be whatever your imagination can conceive (including Babakos) but in fact we can consider without loss of generality that they are only (bi)valuations. It is less poetical but simpler.

Jaśkowski's logic

The first idea is the following. We consider sets of *classical* valuations. Let us call any such a set a *Jaśkowski frame.* Then in a Jaśkowski frame we say that a formula is true iff it is true in *at least one* valuation of the frame. With this we define a Jaśkowski's logic, by saying that a formula is valid (or is a theorem) iff it is true in any Jaśkowski frame, and we define the consequence relation accordingly. This logic is a full paraconsistent classical self-extensional, but it is non adjunctive.

It is based on a very intuitive idea which is the main idea of Jaśkowski's *discussive* logic: when you have a group of people discussing, you can say that something is true if at least one of them thinks it is true. It sounds a little bit chaotic, but it is very democratic (maybe too much). Apparently it should yield to something quite different from classical logic, but it yields to something surprisingly close since Jaśkowski's logic has the same theorems as classical logic. Moreover we have: $a \wedge \neg a \vdash b$ but not $a, \neg a \vdash b$.

How can this be understood? $a \wedge \neg a$ is always false in a Jaśkowski frame, because such a frame is a set of classical valuations. Intuitively: any member of the discussion group reasons in a classical way. Therefore if we have a contradiction of the form $a \wedge \neg a$, any individual of any group will deduce anything from it.

Nevertheless we can find a Jaśkowski frame where a formula a is true and its negation $\neg a$ is true. Intuitively: we can have a discussion group of two people, one who thinks that a is true, and the other one that a is false, therefore that $\neg a$ is true, and they can both agree that b is false. At the end we have a frame which validates a and $\neg a$ but not b.

Despite the very intuitive motivation of Jaśkowski's logic, one can wonder if it really works. If one forgets the intuitive idea and concentrate only on the basic mathematical features of its negation, what is the picture? We have on the one hand a negation which has too much properties, which is in fact quite similar to classical negation, and on the other hand a conjunction that has too few properties.

One can maybe improve the situation by generalizing the idea of Jaśkowski frame, by considering sets of non classical valuation (see [53]).

[24] Paraconsistent logics developed using semantics close to possible world semantics have been also presented in [3], [51] and [52].

Molière's logic

Let us examine now another kind of leibnizian paraconsistent logic. Let us call a Molière frame, a set of valuations which is defined in the following way. The conditions for binary connectives are the usual ones. A formula like $a \wedge b$ is true in a Molière frame iff it is true in any valuation of the frame and it is true in a given valuation iff both a and b are true in this given valuation. Now the condition for negation is as follows: in a given valuation of the frame $\neg a$ is false iff a is true in every valuation of the frame.

The idea is also quite intuitive: we are sure that $\neg a$ is false iff we are absolutely sure that a is true. In a case of doubt about a, let us say, to pursue Jaśkowski's metaphor, if there is someone in the discussion group who thinks that a is false, then $\neg a$ can be false too.[25]

This definition of negation is exactly dual to the definition of negation in the possible world semantics for intuitionistic logic (the difference is that we don't consider accessibility relations). As it is known there is a close connection between intuitionistic logic and the modal logic $S4$. So one may expect a connection between Molière's logic and a modal logic. In fact Molière's logic is nothing else than $S5$ itself. In a given valuation v of a Molière frame, $\neg a$ is true iff there exists a valuation w in which a is false. This means that the classical negation $\perp a$, of a is true in w. Therefore $\Diamond \perp a$ is true in v. So the negation of Molière's logic is nothing else than the connective $\Diamond \perp$ of $S5$ (where $\perp$ is classical negation).

Although Molière negation enjoys some nice properties (it is self-extensional, obeys several de Morgan laws, etc.) and have an intuitive interpretation, it has some drawbacks. For example it is a full paraconsistent logic. Anyway maybe Molière negation can be considered at the present time the best paraconsistent negation. What is funny is that people were trying to construct paraconsistent negations and there was one pretty ready just nearby. But at the end logicians are all funny *bourgeois gentilhommes*, aren't they?[26]

Waiting for nice paraconsistent negations

What can we conclude combining our philosophical investigations and our little tour in the land of paraconsistency?

Certainly until now, no paraconsistent negations having "nice" features have been presented. By "nice", we mean having interesting mathematical properties together with a coherent intuitive interpretation. That does not mean that there are no such things, but at least they have not been discovered yet.

The present investigations do not permit one to be very optimistic about the chance to discover such things, since many classical techniques of mathematical logic, such as logical matrices, possible world semantics, sequent calculus, etc., have been applied - not in a real systematic way, it is true - without success.

But we can still hope. Maybe an entirely new technique must be developed to generate the challenging objects paraconsistent negations are.

[25] It is clear that Molière's logic could also be considered as a formalization of Jaśkowski's idea.

[26] On Molière's logic, see [54], [14]. We would like to thank Claudio Pizzi with whom we had the opportunity to discuss Molière's logic in his castle of Copacabana.

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