Preface of an Anthology of Universal Logic From Paul Hertz to Dov Gabbay[1]

This book is a retrospective on universal logic in the 20th century. It gathers papers and book extracts in the spirit of universal logic from 1922 to 1996. Each of the 15 items is presented by a specialist explaining its origin, import and impact, supported by a bibliography of correlated works. Some of the pieces presented here, such as "Remarques sur les notions fondamentales de la méthodologie des mathématiques" by Alfred Tarski, are for the first time translated into English.

Universal logic is a general study of logical structures. The idea is to go beyond particular logical systems to clarify fundamental concepts of logic and to construct general proofs. This methodology is useful to understand the power and limit of a particular given system. Lindström's theorem is typically a result in this direction: it provides a characterization of first-order logic. Roughly speaking, Lindström's theorem states that first-order logic is the strongest logic having both the compactness property and the Löwenheim–Skolem property (see details in Part 10). Such a theorem is concerned not only with first-order logic but with other nearby possible logics. One has to understand what these other possible logics are and be able to compare them with first-order logic. In short: one has to consider a class of logics and relations between them. Lindström's theorem is a result in favor of first-order logic, but to claim the superiority of this logic one must have the general perspective of an eagle's eye. Moreover Lindström's theorem favors first-order logic within a limited galaxy of possible logics. At a more universal level, things change. One may want to generalize Lindström's theorem to other galaxies, such as the galaxy of modal logics (about such generalization see e.g. [5]). In order to do so, we need a clear understanding of what Lindström's theorem exactly depends on.

Comparison of logics is a central feature of universal logic. The question of *translation* of a logic into another one is directly connected to it. This topic is especially treated in Part 4. Gödel has shown that it is possible to translate intuitionistic logic into a system of modal logic and more surprisingly classical logic into intuitionistic logic, a surprising result since in some sense intuitionistic logic is strictly weaker than classical logic (among other things, the excluded middle holds in the latter but not in the former). There are other cases of such a paradoxical situation, e.g. the occasion of a logic weaker than another logic which can however be translated into its weaker sister (see [7] and [24]). This is

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a phenomenon similar to Galileo's paradox showing that we may have a one-to-one correspondence between a set and one of its proper subsets. Galileo's paradox is cleared up by showing that there are two concepts corresponding to two contiguous but distinct notions. Clarification of concepts is also a device to solve a translation's paradox. We must have a good definition of what the *strength* of a logic is, understanding that there is not only one way to compare logics, in particular that there are different non-equivalent ways to translate one logic into another. It is important to point out that we cannot solve this problem just by straightforwardly importing concepts from other part of mathematics, thinking they will do the job in the logic realm. Intuitionistic logic is not a *sublogic* of classical logic in the same sense that rational arithmetic is a *subalgebra* of real arithmetic. One has to avoid the famous sufism of Nasruddin: a man at night looking for his key, not where he lost it, but under the light of a not so nearby lamp-post.

Another paradox appears when *combining* classical logic with intuitionistic logic. They may collapse into the same logic, contrary to the expectation of the theory of combination of logics: to get the smaller conservative extension of both logics preserving their own idiosyncrasies (see [35] and [14]). An opposite paradox in combination of logics is the copulation paradox: instead of having less, we have more: e.g. by putting together the logic of conjunction with the logic of disjunction, we may get distributivity (see [8] and [9]). To avoid these paradoxes we must develop a good theory of combination of logics and to do so we must find the right concepts. With these paradoxes the logician is confronted with some particular cases that must be taken into account and analyzed to build a nice abstract theory. So in some sense logic is an empirical science, in the sense that the logician is facing some objective phenomena that cannot be dropped, whatever their private reality is. As it is common in the history of mathematics, first particular cases are studied and then the level of abstraction rises. This is typically what has been happening in the theory of combination of logics. First logicians were combining modal logics, and then they started to develop a general theory of combination of logics, in particular Dov Gabbay with his pivotal concept of fibring (see Part 15).

But the abstraction rise is not necessarily progressive, there are also some radical jumps into abstraction. In logic we can find such jumps in the work of Paul Hertz on *Satzsysteme* (Part 1) and of Alfred Tarski on the notion of a *consequence operator* (Part 3). What is primary in these theories are not the notions of logical operators or logical constants (connectives and quantifiers) but a more fundamental notion: a relation of consequence defined on undetermined abstract objects that can be propositions of any science but also data, facts, events. Probably Hertz and Tarski did not directly think of all possible interpretations of such abstract objects. When performing jumps into abstraction, we cannot foresee the true depth and breadth of the realm which is being opened. In universal logic, consequence is the central concept. But this consequence relation is neither syntactical (proof-theoretical), nor semantical (model-theoretical). We are beyond the dichotomy syntax/semantics (proof theory/model theory). This level of abstraction is the highest vertex of an upward pointing triangle with syntax and semantics as base angles. It is the crucial point of the completeness theorem; by reaching it we are led to its trivialization, following Wójcicki's way of speaking. In the original work by Hertz and Tarski this is not so clear, and one may be confused by the fact that Hertz's name is rather connected with proof theory due to its influence on Gentzen's work, and that Tarski's name is rather connected with semantics due to his work on truth and model-theory. But Hertz's original work is not so proof-theoretical, as shown by its later development in a structuralist perspective by Arnold Koslow [25]. It is also worth recalling that in his first paper Gentzen stays at Hertz's abstract level, proving an abstract completeness theorem (this is discussed in Part 14). Concerning Tarski, it is important to clearly distinguish his work on consequence operators from his work on model theory although there is a connection between the two explained in Dana Scott's paper presented in Part 12.

When we consider a logic as a structure with the consequence relation as the central concept, a relation which can be defined in many different ways but that can also be considered independently of any particular specification, we can say that we are at the level of *abstract logic*. The terminology "abstract logic" was much used by Roman Suszko. In the 1950s Suszko developed his work with Jerzy Łós defining the consequence operator over an absolutely free algebra, leading to the notion of a structural consequence operator, in the spirit of Lindenbaum representation theorem of logical matrix theory (Part 7). At the end of the 1960s Suszko, along with Donald Brown and Stephen Bloom, came back to a more abstract setting that he explicitly called "abstract logic", considering a consequence relation over an undetermined abstract algebra (Part 11), a level of generalization not as high as the one of Tarski's first framework of consequence operator, but higher than the one of abstract model theory where the expression "abstract logic" is sometimes also used. Suszko and other people in Eastern Europe had the idea that logic was part of universal algebra, a mathematical trend highly popular in the East. There was an assimilation of abstract logic with universal algebra, connected with a broader assimilation, that of universal algebra with mathematics as a general theory of structures. In this context the differences between Boolean algebras, lattices and any mathematical structures is just a question of level of abstraction. There may be confusion in this mixture, such as when many years ago the word "structure" was used as synonymous with "lattice" by Glivenko (see [20] and [17]).

Universal logic can be defined as a general study of logical structures in the same way that universal algebra is a general study of algebraic structures. The word "universal" in "universal logic" is used according to this analogy: as in universal algebra, universal logic is not a universal system, but rather a universal systematization. Universal algebra is not one algebraic system encompassing everything, but a bunch of global concepts allowing us to unify the treatment of the multiplicity of algebraic structures. These concepts were mainly put forward by Garrett Birkhoff in the 1930s (see [10] and [11]). The central concept is the concept of abstract algebra defined by Birkhoff just as a set with a family of operators. The spirit of universality is the same in universal logic and universal algebra, but these two fields are different because a logic structure is not necessarily an abstract algebra. Reduction of logic to algebra can be developed through algebraization of logic, which can mean both the reduction of logical structures to algebraic structures and the application of algebraic methods to logic. Generally the former is seen as a first step towards the latter. But although it is interesting to make a connection between logic and algebra, there is no good reason to think that logic reduces to algebra. There are indeed logical structures that cannot be algebraized (see [6]). One may also apply other mathematical methods and tools to develop logic, for example topology. The initial Tarskian concept of consequence operator is in fact closer to topology than algebra. But the very idea of universal logic is that logical structures are different from other mathematical structures and that more generally logic is different from other parts of mathematics.

To have a deeper understanding of this, we have to think at the level of a general theory of mathematical structures. One may think of category theory. But category theory will not, right at the start, clarify what a logical structure is. And to apply category theory to logic in the perspective of an alternative foundation of mathematics, as it was done by Lawvere [26], is not the same as developing a study of categories of logical structures [2]. In this latter case we consider logical structures encompassing classical and non-classical logics, working with a pre-categorial vision of the concept of logical structures based on the reality of the huge variety of logical systems. A famous categorization of logical structures is due to the late Joseph Goguen, originator of the concept of *institution* (see Part 13). What is fascinating is that the origin of this trend of general abstract nonsense is computer science, a very concrete and applied science. Goguen described the situation as follows: "The enormous and still growing diversity of logics used in computer science presents a formidable challenge. One approach to bringing some order to this chaos is to formalize the notion of *a logic* and then systematically study general properties of logics using this formalization, including the representation, implementation, and translation of logics. This is the purpose of the theory of institutions, as developed and applied in a literature that now has hundreds of papers." [21]

To understand what a logical structure is, it is worth having a look at Bourbaki's monumental work. Bourbaki was the first to develop a general theory of mathematical structures. To avoid misunderstandings, one must distinguish Bourbaki's informal theory expressed in his 1948 paper *L'Architecture des mathématiques* [12] and the formalization of it presented in a chapter of the 1954 book *Théorie des ensembles* entitled *Structures* [13]. The defect of this dated set-theoretical formalization cannot be used as a fatal argument against the informal theory. More than anything it is important to remember that Bourbaki was the first to develop a general theory of mathematical structures and to consider *morphism* as a central concept of mathematics (see [4] and [16]). From the bourbachic viewpoint it is clear that mathematical structures do not reduce to algebraic structures. According to Bourbaki there are three distinct classes of *mother structures* from which we can reconstruct mathematics, mixing them thus generating *cross structures*. These fundamental classes of structures are structures of order, algebraic structures and topological structures. Bourbaki was not excluding the existence of other mother structures, as recalled by Jean Porte, who tried to develop logic in the bourbachic spirit—his PhD advisor was René de Possel, one of the founding members of Bourbaki. Porte, in his very interesting book *Recherches sur la théorie générale des systèmes formels et des systèmes connectifs* published in 1965, studies different classes of logical structures taking into account and including the recent developments of the Polish school on consequence operators and logical matrix theory (see Part 9).

Porte clearly states that he is not doing *metamathematics*, explaining that the logical systems he is studying are not exclusively describing mathematical reasoning. For him logic is mathematical, but not necessarily about mathematics. The expression "mathematical logic" is highly ambiguous because it can mean the *logic of mathematics* or a *mathematical study of logic* (this ambiguity was already noted by Zermelo in 1908, see [30, p. 320]). First-order logic can be seen as a combination of the two. But these two orientations may be quite different and they indeed were different in the history of modern logic. They can be distinguished in a broad outline using the opposition between the *Boolean way* and the *Fregean way*. Boole was using mathematics to understand the laws of thought, and these are not only concerned with mathematical thinking. Boole had a general perspective on reasoning, as Aristotle had: syllogistic is about any kind of reasoning. But syllogistic is neither mathematical in a Boolean sense—it does not use mathematics to describe reasoning—nor in a Fregean sense—it does not give an accurate description of mathematical reasoning. Frege's main interest was to describe reasoning of arithmetics, for doing that he was not using mathematics, but some two-dimensional graphism with a cryptic name: *Begriffsschrift*. Such ideography is not more mathematical than musical notation. A similar tendency—the use of a non-mathematical technique to describe mathematical reasoning—can be found with Peano's pasigraphy and with Whitehead's and Russell's *Principia Mathematica*. Jean van Heijenoort puts these tendencies in the same basket: according to him the three of them have the feature of a *lingua characteristica* (see [23]). We can consider first-order logic as a mix: rules of syntax (construction of the language) and rules of proofs (proof theory) are generally closer to a *lingua characteristica* orientation, but mathematics is extensively used for the semantics developed in model theory. This mix was described by Chang and Keisler in their classical book [15], by the equation *model theory* $=$ *logic* $+$ *universal algebra*. In this equation "logic" may be interpreted as logic syntax and "universal algebra" as a class of mathematical structures (Chang and Keisler are here under the influence of the reduction we were mentioning: *structures* $\subseteq$ *algebras*).

A mix also appears at another level qualified as *mathematics of metamathematics* by the Polish duo Rasiowa/Sikorski [33]. This can be considered as synonymous to *algebraic logic*. Such use of mathematics is different from the use of mathematics at the semantic level. This can be understood through the example of classical propositional logic: its semantics is the Boolean algebra on {0, 1} but classical logic can directly be considered as a Boolean algebra by factoring it. These are two different methodologies, the first is mainly due to Post [32] and the second, more than 10 years later, to Tarski (not to Lindenbaum as often erroneously stated). This is the second methodology which is now usually qualified as algebraization of logic. Paul Halmos has generalized this methodology to first-order logic popularizing the expression "algebraic logic" (see [22]), but this terminology was introduced first by Curry and with a different meaning more connected with universal logic.

Haskell Curry was the last PhD student of Hilbert and tried to systematically develop the formalist approach, both at the philosophical and metamathematical levels. But finally, as pointed out by Seldin [34], his approach would be better qualified as structuralist. In the present anthology Seldin translates and comments extensive extracts of Curry's monograph *Leçons de logique algébrique* (Part 6). This book is not very well known and has never been translated into English. It was published in 1952, eleven years earlier as a more famous book by Curry, *Foundations of mathematical logic* [19]. As indicated by the title of this later book, Curry is interested in the foundations of logic, not in the foundations of mathematics. This difference may not be immediately caught because: 1) someone may not pay attention to the way the three notions foundations/mathematics/logic are combined, understanding Curry's title as synonymous with logical foundations of mathematics; 2) if we consider that mathematical logic is mainly concerned with mathematical reasoning, foundations of mathematical logic has to do with foundations of mathematics. But Curry, like Porte and the Poles, is not interested only in mathematical reasoning, he has also interest in many systems of logic describing reasoning concerning other fields, like quantum logic. His 1952 monograph is a study of many different systems of logic and to do so he develops a general framework, using some mathematical tools, in particular mathematical structures that are more or less algebraic structures.

In the 1950s Curry was the first to pay attention to the work of Saul Kripke. Kripke as a teenager wrote to him and they maintained a correspondence during a couple of years (see

[19, p. 240, 243, 250, 306]). Curry had much interest in modal logic, however, due to his formalist background he had been working more on its proof theory. But Curry was open minded, interested in all aspects of logic, having an impressive knowledge of what was going on in logical research, as we can see through the excellent bibliography of [19]. It is worth emphasizing that Kripke also had broad interests, he was working in many different logical systems other than modal logic, as we can see through Curry's book, such as multiple conclusion sequent systems for intuitionistic logic (p. 306). Kripke was considering logic in a general perspective but probably he was not aware that by developing a semantic framework for modal logics, he would provide a universal tool whose applications go far beyond the field of modal logic. Modal logic is a field of research that is directly connected to universal logic: there are many different systems of modal logic and it has been natural to develop a general theory of the class of modal logics. This theory gives good indication of how universal logic can be developed, giving hints for the study of other classes of logics and for systematization of logical structures in general. For example it is obviously useful to find the general formulation of the completeness theorem beyond all variations of Kripke structures, or the general techniques for combining them. Kripke structures can also be used to deal with many systems of logics other than modal systems: intuitionistic logic, relevant logic, paraconsistent logic. Moreover they are a powerful tool for making links between propositional logics and higher-order logic, in particular reducing fragments of first-order logic into propositional modal systems, work especially developed by Johan van Benthem (see e.g. [1]) who presents Kripke's work in Part 8. We may introduce the expression *Kripke logics* to name the class of logics that can be defined using Kripke structures. This class does not reduce to modal logic. And vice versa the class of modal logics is not included in the class of Kripke logics.

It is important to recall that the first semantics for modal logic is due to Łukasiewicz, it is a three-valued matrix semantics [28]. Later on Łukasiewicz developed a four-valued matrix semantics [29]. Matrix semantics does not reduce to modal logic. The tendency in fact, due to the awkwardness of Łukasiewicz's systems and Dugundji's negative result about characterization of S5 by finite matrices, is to consider that we have here two disjoint classes of logics: on the one hand the class of logics definable using logical matrices that can be called *truth-functional logics* (rather than many-valued logics, see [31]), on the other hand the class of modal logics. This does not mean that the class of truth-functional logics and Kripke logics are disjoint. For example classical propositional logic is a truth-functional logic and also a Kripke logic, even if the Kripke semantics for it is rather trivial (one may also argue that the truth-functional semantics of classical propositional logic is trivial compared to much more complex truth-functional semantics). Łukasiewicz used matrix semantics in a philosophical perspective, but Tarski saw it rather as a universal tool and it was developed as such in the Polish school, in particular by Lindenbaum. The work of Lindenbaum on logical matrices has been published mainly through Łos's monograph [27]. But logical matrices is not a pure Polish product, it was developed also by Emil Post presenting bivalent matrices for classical propositional logic [32] and immediately generalizing the technique (Post was born in Poland, but grew up in the USA) and by Paul Bernays. In Part 2 Bernays's work is presented: using many-valued matrices, not to develop a particular non-classical logical system as Łukasiewicz did, but to analyze the independence of axioms for classical propositional logic.

As pointed out by Suszko [36], it is possible to provide a bivalent semantics for Łukasiewicz's logic L3. This apparent paradox is cleared up when we know that this bivalent semantics is not truth-functional. A general theory of non truth-functional bivalent semantics has been developed by Newton da Costa and his school, the *theory of bivaluations* (see Part 14). We can call *da Costa logics* logics that can be defined by non truth-functional bivalent semantics. Da Costa's idea was that non truth-functional bivalent semantics is a universal tool in the sense that any logic can be defined using it; in other words, the class of da Costa logics is the universal class of logics. This is a contestable claim because we can argue that some structures that can rightly be called logics are not definable with this tool. In fact it is doubtful that there exists any tool that can be used to define all logics. There is no such logic wand. But anyway the theory of bivaluations is very interesting for at least two reasons. Firstly it breaks the illusion of the completeness theorem as a magical result connecting two worlds, on the one hand a world made of strings of symbols, on the other hand a world of flesh and blood models. In the theory of bivaluations, a model is just a set of formulas. Secondly the theory of bivaluations is used to study many non-classical logics. In fact da Costa was led to this theory trying to provide semantics for his systems of paraconsistent logic [18]. As it is known, da Costa is the main promoter of paraconsistent logic, and contrary to other people like Asenjo [3] or Jaśkowski developing paraconsistent logic in a rather Sufist way, using respectively the lamp-post light of matrix semantics and modal logic, da Costa invented a new tool having broad applications, being a new light for the universal dimension of logic. Using da Costa's theory of bivaluations it is possible to develop *paranormal logics*, logics which are both paraconsistent and paracomplete. It definitively shows that the foundations of logics lay far beyond some principles such as the principle of non-contradiction or the excluded middle. At the abstract level everything is possible, concepts are elaborated that can be used to develop tools that can be applied to many concrete situations.

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