Three Sisters : Philosophy, Mathematics and Logic

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Abstract

We examine here the relations between mathematics, philosophy and logic. We start with an analysis of the motto at the entrance of Plato's academy and the meaning of Plato's dialectic, inspired but not limited to mathematics. We go on giving a brief overview of logic from Aristotle to Tarski including the rejection of syllogistic by Descartes and Pascal. We then discuss the many different and sometimes contradictory relations between these three sisters in modern times: mathematics used to develop an advanced theory of reasoning, logic used to have a better understanding of mathematics in a deep philosophical sense, logic used as a toy for philosophical argumentations. We end by a discussion about truth, a shared and disputed notion.

1. In the Beginning, the Academy

According to the legend, at the entrance of the Academy, Plato put the following sentence: *let no one inapt to geometry come in* (about this translation and the history of this motto see Suzanne 2004). Nowadays Plato is considered as a philosopher and geometry as part of mathematics and from this point of view one may interpret this sentence as claiming that before starting to study philosophy one has to know mathematics, something which seems a bit strange. Today there is no obligation to know mathematics to enter a philosophy institute and generally philosophers know very few about mathematics and don't like it. This is not a recent phenomenon: philosophers like Kant and Hegel had a very poor knowledge of mathematics, limited to 2+2=4 or at best 7+5=12. However before them people like Descartes, Pascal and Leibniz were both philosophers and mathematicians. Does this mean that there was a turning point in the 18th century? This is not so clear. On the one hand before Descartes someone like Aristotle had few interest for mathematics and on the other hand after Descartes, a philosopher like Bertrand Russell had great interest for mathematics

At the time of ancient Greek, neither mathematics, nor philosophy were names for fields of investigations. The academy of Plato was not a school of philosophy in the present sense of the word "philosophy", it was a place to learn many different things like music and astronomy. It can be considered as the first university. Modern universities have kept the general platonic spirit in the sense that they are not practical schools to learn a technique to earn a living. Nowadays the word "academy", putting aside some degenerated cases, is mainly used in two different senses: one inherited from the Renaissance, meaning a prestigious and honorific institution related to arts and/or science, the other one closer to the original sense of the word, and rather expressed through an adjective: "academic" meanings related to university.

Plato, the originator of the academic world in this sense, is considered mainly as a philosopher, but today philosophy is just a marginal subfield in the universities. Nevertheless the word philosophy is hidden in the highest degree one is getting at universities: a PhD, which means Doctor of Philosophy. If someone is PhD in Meteorology, this means he is Doctor of Philosophy in Meteorology. But it is possible also to be PhD in Philosophy, which means

Doctor of Philosophy in Philosophy. To explain this redundancy we have to understand here the two occurrences of the word "philosophy" in two different ways. The first meaning can be interpreted as knowledge in general. A doctor of philosophy in X is someone who has knowledge in X. But what is the meaning of the second occurrence? What do students learn in a course of philosophy at university? They learn things like history of philosophy, esthetics, ethics, metaphysics, epistemology and sometimes logic, but not mathematics.

To enter a department of philosophy at a university one does not have to pass a selective math exam. But students learning philosophy at university have learned mathematics before, like in fact all students entering university, because mathematics is an essential part of the teachings in primary and secondary schools. In this sense we can say that the platonic perspective has been preserved.

But can we assimilate the mathematics learned at school with the "aptitude of doing geometry"? School mathematics is basically calculation: how to make an addition, a multiplication solving an equation, etc. If mathematics is understood in this way, it sounds strange indeed to require the study of mathematics before studying something like esthetics, although it may be useful for other fields at university, like physics.

When saying *Let no one inapt to geometry come in* Plato probably didn't think of this kind of mathematics and the fact that he used the notion of "geometry" rather than the notion of "number", despite the fact that as a neo-Pythagorean he was found of numbers, suggests that he was referring to reasoning rather than calculation. Greek geometry may be considered as the symbol of the hypothetico-deductive method, a method that Plato was considering as a model for what he called dialectic. Dialectic is similar to geometry in the sense that it goes step by step by a certain systematic method that Plato illustrated with dichotomy. On the other hand Plato emphasized that the difference between dialectic and geometry is that dialectic goes beyond hypothesis. Here is a famous excerpt at the end of the book VI of the *Republic* (511): "And when I speak of the other division of the intelligible, you will understand me to speak of that other sort of knowledge which reason herself attains by the power of dialectic, using the hypotheses not as first principles, but only as hypotheses-- that is to say, as steps and points of departure into a world which is above hypotheses, in order that she may soar beyond them to the first principle of the whole; and clinging to this and then to that which depends on this, by successive steps she descends again without the aid of any sensible object, from ideas, through ideas, and in ideas she ends."

How to practice dialectic is not really clear. Anyway if we identify what Plato was calling dialectic with what was later on called philosophy, identification which is quite natural and one of the reason why Plato is considered mainly as a philosopher, we see that in this sense philosophy is rather a method than a field of study. This meaning of philosophy has in fact being lost in the sense that philosophy as a study of esthetics, ethics, politics, history of philosophy is rather object-oriented than method-oriented. This is maybe due to the fact that Plato and his successors were not able to present a serious methodology. The word dialectic was later deformed by Hegel and Marx not improving the situation. However we may find nowadays this method-oriented approach in philosophy in teachings like logic or critical thinking.

2. Logic from Aristotle to Tarski

Critical thinking is something which is taught mainly in American universities and this kind of courses have a tendency to replace standard courses of logic in philosophy and/or humanities curriculums. A standard course of logic for philosophers is generally the presentation of classical propositional logic and elementary first-order logic. Some techniques like truth-tables, natural deductions or tableaux are presented, and students are asked to translate sentences of ordinary language into formulas using symbols like $\forall$, $\exists$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, sometimes $\Diamond$, □. One may wonder if such kinds of techniques are useful as a methodology to develop thinking. By contrast a course of critical thinking is generally an informal analysis centered on fallacies, ranging from the story of cheap horses that are expensive to the latest absurdity in the discourse of a famous politician. This duality *logic techniques / analysis of fallacies* is in fact very similar to Aristotelian logic: on the one hand Aristotle did propose a technique like syllogistic, which was quite useless as a basis for thinking, and on the other hand he presented an analysis of the many wrong ways of reasoning (in his book *On Sophistical Refutations*).

Following Aristotle, logic was considered as the basis for superior studies, it was qualified as an "instrument" - *Organon* – by his followers. What is the relation between Aristotle's logic and Plato's dialectic? Even if we consider that the logic of Aristotle does not reduce to syllogistic, the main difference is that Aristotle's approach is rather an analysis of how we reason than a methodology for thinking, despite that it can sometimes appear as quite normative. One may consider that Plato proposed only a very rough methodology with his dialectical dichotomic method, cutting everything in two, but Aristotle didn't propose any methodology which can fruitfully be applied. No mathematician has ever used syllogistic to prove a theorem, it has also not be efficiently used to prove the existence of God or that human beings derived from monkeys. Descartes and Pascal were conscious of this fact and strongly criticized Aristotle's logic. This was the start of a renewal for science and philosophy.

Descartes wrote a famous book called *Discourse on the method* (1637), whose full title is *Discourse on the Method of Rightly Conducting One's Reason and of Seeking Truth in the Sciences. Plus the Dioptric, Meteors, and Geometry, Which are Essays in this Method*. This book presents a general method and then applications of it. This method can be summarized in the following table:

DESCARTES 4 PRECEPTS	
Clarity	Never to accept anything for true which I did not clearly know to be such; that is to say, carefully to avoid precipitancy and prejudice, and to comprise nothing more in my judgment than what was presented to my mind so clearly and distinctly as to exclude all ground of doubt.
Division	To divide each of the difficulties under examination into as many parts as possible, and as might be necessary for its adequate solution.
Ascension	To conduct my thoughts in such order that, by commencing with objects the simplest and easiest to know, I might ascend by little and little, and, as it were, step by step, to the knowledge of the more complex; assigning in thought a certain order even to those objects which in their own nature do not stand in a relation of antecedence and sequence.
Exhaustivity	To make enumerations so complete, and reviews so general, that I might be assured that nothing was omitted.

These four simple precepts should according Descartes replace the intricate and useless syllogistic. It is important to emphasize that these precepts help to think systematically but do not appear as a system like syllogistic. They are favoring analysis but they are practical indications for developing our thoughts, not an analysis of how we are rightly or wrongly reasoning.

These precepts do not appear has inspired by mathematics. In the *Discourse on the Method* first is described the method based on these precepts and then the method is applied to mathematics, more explicitly to geometry. These four precepts are not derived from an esoteric aptitude developed by mathematicians acquainted to deal with mysterious abstract objects, but are based on good sense, which is "of all things among men, the most equally distributed", as Descartes famously claimed in the opening of his *Discourse*.

Blaise Pascal's criticism of syllogistic is along the same straight line as Descartes' one but the methodology he is proposing is more Platonic; it is directly connected with geometry. It is presented in his booklet *On the Geometrical Spirit* (1657) and can be summarized as follows.

PASCAL 8 RULES	
Rules for Definitions	Not to undertake to define any of the things so well known of themselves that clearer terms cannot be had to explain them.
	Not to leave any terms that are at all obscure or ambiguous without definition.
	Not to employ in the definition of terms any words but such as are perfectly known or already explained.
Rules for Axioms	Not to omit any necessary principle without asking whether it is admitted, however clear and evident it may be.
	Not to demand, in axioms, any but things that are perfectly evident of themselves.
Rules for Proofs	Not to undertake to demonstrate any thing that is so evident of itself that nothing can be given that is clearer to prove it.
	To prove all propositions at all obscure, and to employ in their proof only very evident maxims or propositions already admitted or demonstrated.
	To always mentally substitute definitions in the place of things defined, in order not to be misled by the ambiguity of terms which have been restricted by definitions.

This framework proposed by Pascal is a refinement of the axiomatic method of Greek geometry. It has the ternary articulation: *Definitions*, *Axioms*, *Proofs*. Tarski was strongly influenced by Pascal, as he explicitly said in his lecture presented in Paris in 1936 (published as Tarski 1937), which appears also as Chapter 6 of his book *Introduction to Logic and the Methodology of Deductive Sciences* (1936). And we can say that generally modern logic was developed in this spirit, the difference being the rise of abstraction as initiated in particular by Hilbert with his *Foundations of Geometry* (1899) and systematically developed by Tarski with Model Theory.

Pascal put emphasize on definitions and proofs but he had the idea that it is not possible and/or necessary to define and prove everything. He defends a “middle way“, typical of his philosophy: “This order, the most perfect of any among men, consists not at all in defining every thing or in demonstrating every thing, nor in defining nothing or in demonstrating nothing, but in adhering to this middle course of not defining things clear and understood by all mankind, and of defining the rest; of not proving all the things known to mankind, and of proving all the rest. Against this order those sin alike who undertake to define everything and to prove every thing, and who neglect to do it in those things which are not evident of themselves.“

Pascal argues that it is quite absurd wanting to define *human* or *number*: “For there is nothing more feeble than the discourse of those who wish to define these primitive words … because these terms so naturally designate the things that they mean, to those who understand the language, that their elucidation would afford more obscurity than instruction.“ But at the same time Pascal had the idea that this is not only absurd but impossible : “This method would certainly be beautiful, but it is absolutely impossible; for it is evident that the first terms that we wished to define would imply precedents to serve for their explanation, and that in the same manner“. This impossibility is connected to a general philosophical mood according to which human beings are limited, which has many variations, to mention just two famous ones: Kant and Heidegger. This is opposed to some views like the ones of Hegel or Plato.

The contrast with Plato is clear because Pascal considered geometry as the zenith of human mind: “But it is first necessary that I should give the idea of a method still more eminent and more complete, but which mankind could never attain; for what exceeds geometry surpasses us; and, nevertheless, something must be said of it, although it is impossible to practice it“.

3. Mathematical Logic and Mathematical Philosophy

Modern logic has in some sense reinforced the place of logic in the philosophy curriculum, but at the same time it has revealed the lack of interest for mathematics of modern philosophers. The traditional Aristotelian logic had nothing to do with mathematics. Modern logic is related with mathematics in two important ways: mathematics is used as a method to develop the theory of reasoning and it investigates mathematical reasoning. The expression "mathematical logic" is ambiguous because it can mean either "mathematized logic" or "the logic of mathematics". But these two trends can be independent. In fact Boole was using mathematics to develop a general theory of reasoning, not especially a theory of mathematical reasoning, on the other hand Frege was mainly interested in a theory of mathematical reasoning (arithmetic) and he didn't really use mathematics to develop such a theory. We still find these two directions in the XXth century: there are a lot of algebraic logics not dealing with mathematical reasoning, and people dealing with mathematical reasoning are using a "formalism" that mathematicians don't necessarily consider as mathematics.

Investigation of mathematical reasoning by logicians looks rather philosophical to mathematicians. Bertrand Russell besides promoting the expression "principles of mathematics" also used the expression "mathematical philosophy", which is quite ambiguous. Someone may think that "mathematical philosophy" is the application of mathematical methods to philosophical thinking. But Russell didn't use this expression in this sense. Here is how he describes what is *mathematical philosophy*: "The other direction, which is less familiar, proceeds, by analyzing, to greater and greater abstractness and logical simplicity; instead of asking what can be defined and deduced from what is assumed to begin with, we ask instead what more general ideas and principles can be found, in terms of which what was our starting-point can be defined or deduced. It is the fact of pursuing this opposite direction that characterizes mathematical philosophy as opposed to ordinary mathematics."

One may interpret this mathematical philosophy as "philosophy of mathematics" and in fact in other languages like Portuguese the book of Russell (1919) has been translated as *Introduction to the Philosophy of Mathematics.* This is not necessarily wrong but it can be misleading. Because when one talks about philosophy of X, this can be some comments and analysis of X without proper interaction. Nowadays there are philosophies of almost everything: philosophy of music, philosophy of sport, etc. and there is even a tendency to reduce philosophy to "philosophy of".

But mathematical philosophy in the sense of Russell is much more than philosophy of mathematics, it is a way to go deeper and deeper in the root of mathematics, similar to what Plato was considering as dialectic. But Plato was not stressing the application of dialectic to mathematics, and on the other hand the mathematical philosophy of Russell is a kind of philosophy centered on mathematics.

The connection between the mathematical philosophy of Russell and logic has two different aspects (not necessarily equivalent): the idea that basic concepts of mathematics can be reduced to logical notions (*logicism*), the very *analysis* of the basic concepts of mathematics is fundamentally "logical" (in an informal Cartesian sense). These two aspects of mathematical philosophy are related to *analytic philosophy*, of which Russell can be seen one of main originators, but analytic philosophy does not reduce to mathematics and also does not promote a reduction of philosophy to logical notions. It promotes analysis to think about anything, using logical methods in different ways.

4. Analytic Philosophy, Logic and Mathematics

There is no doubt that analytic philosophy is linked with logic, what is not clear is the linkage between analytic philosophy and mathematics. Many analytic philosophers are interested in questions related to philosophy of mathematics, in particular to the question of reduction of mathematics to logic, but also to the question of the nature of mathematical objects. On the other hand much of the time analytic philosophers when applying logical methods to deal with different questions are not really using mathematics, their approach looks more like a formal game using some symbols aliens of natural language, but this is not a guarantee that they are doing mathematics. The mathematical logician Paul Halmos makes the following comment about how to write mathematics: "the best notation is no notation" (Halmos 1985).

Analytic philosophers like very much "arguing". The ambiguity of this word in English, which can mean quarreling, perfectly reflects the sophistic dimension of their approach, opposed to mathematical thinking: a mathematician will not say that he is arguing, he will rather say that he is proving. However we can agree with Plato, saying that philosophy does not reduce to mathematical proofs, it goes beyond. But Plato was aware that dialectic may degenerate in quarrels: “And is it not one chief safeguard not to suffer them to taste of it (dialectic) while young? For I fancy you have not failed to observe that lads, when they first get a taste of disputation, misuse it as a form of sport, always employing it contentiously, and, imitating confuters, they themselves confute others. They delight like young dogs in pulling about and tearing with words all who approach them“ (*Republic* VII, 539).

It is interesting at this stage to compare the doggy style of some philosophers using logic, selling and buying arguments, with a more tantric approach to reasoning practiced in mathematics. Here is how it is pictured by the bourbachic master André Weil: “Every mathematician worthy of the name has experienced, if only rarely, the state of exaltation in which one thought another as if miraculously, and in which the unconscious (however one interprets the word) seems to play a role … Unlike sexual pleasure, this feeling may last for hours at a time, even for days. (Weil 1991, p.91)

One could say that this “methodology” can also be applied to concepts and objects outside of mathematics and also to mathematics itself. And that would be a good definition of philosophy. Weil also quotes Gauss describing the mathematical activity as follows: “to conceive is a pleasure, but to give birth is painful”. This remembers maieutic, another way to characterize the dialectic practiced by Socrates, son of a midwife.

It is not clear that one has to experience first this method in mathematics but the fact is that it is where it explicitly manifests. And Plato thought that such training was useful to avoid degeneration in quarrelling. This kind of activity is logical in the sense that it promotes reasoning. But this is not a mechanical reasoning based on some logical system. On the contrary this “logical” reasoning can be used to develop logical systems.

Modern mathematics was a way to develop intelligent reasoning at an early age not limited to numbers and/or spatial objects. One of the main promoters of modern mathematics was the Belgian mathematician Georges Papy (1920-2011). He was known as the *King of Potatoes*, but Papy didn’t reduce mathematics to sets as potatoes, he was rather using graphs and was able to present some important reasonings about the infinite using some simple pictures, explaining difficult mathematical proofs with simple pictures understandable by young children. This was the time of Piaget and intelligence. Nowadays it seems that people are afraid of intelligence. What is important is rather to be clever, and people can be so clever that nobody understand what they are talking about, even themselves. As Oscar Wilde used to say: “I am so clever that sometimes I don't understand a single word of what I am saying“.

5. The Music of Reason, the Art of Thinking and Truth

Mathematics has often been assimilated, by contrast to other sciences, to an artistic activity, in particular music. The book of another bourbachic master, Jean Dieudonné, who liked to play piano, has nicely been translated into English as *Mathematics - the music of reason*.

It is true that logic has also been called an “art”: the famous logic bible of the Port-Royal logicians Arnauld and Nicole is called *The art of thinking* (1662). This is a good title, but there is an ambiguity here, because “art” was in the past often used to mean “technique”. Then we may be back to some kind of mechanization of thought, or worse, a collection of tricks, like in some books of self-help – see the recent best seller *The art of thinking clearly* by Rolf Dobelli. This sense of “art” is opposed both to art as beauty and to thinking oriented towards wisdom and truth.

But what is truth? Mathematicians barely use the word, much less than philosophers and logicians. They certainly believe that mathematics is closely connected to truth in many different ways, but they don’t take this notion as an object of study. This has been done by logicians in many different ways. Logic as foundations of mathematics, or metamathematics, tries to explain the basis of mathematical truth. This exploration has led to various kinds of relativitazions: nowadays it seems difficult to argue that 2+2=4 is an absolute truth. Arithmetic has been axiomatized and 2+2=4 depends on some axioms, it is not a “logical truth”. The notion of logical truth is based on the notion of truth considered as a mathematical object behaving according to some principles, which themselves are relative. They also depend on a system which can be modified: there are a lot of non-classical logics with different definitions of truth and logical truth (see Beziau 2010b). And Gödel used an argument inspired by the liar paradox to show that arithmetic cannot be “completely” axiomatized. A theorem which was not seriously considered by mathematicians before a result by Paris and Harrington giving an example of independent proposition having a mathematical meaning, different from the artificial Gödel's sentence talking about herself. Independently of Gödel's negative result and the variety of definitions of logical truth, according to modern logic, we have not only truth and logical truth, but six situations that can be described by a metalogical hexagon of opposition (see Beziau 2013).

What are the consequences of all these important logical results about truth for philosophy? This is not really clear, on the one hand we have philosophers talking about truth in a metaphorical way close to poetry, and on the other hand philosophers talking about different "theories" of truth, as if they were scientific theories. But most of the time in the first case we have something which is not so beautiful, and in the second case, something which is not so rigorous. This contrasts with mathematics which is a manifestation of beauty and rigor.

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