MANY 1

A Transversal Imaginative Journey across the Realm of Mathematics[1]

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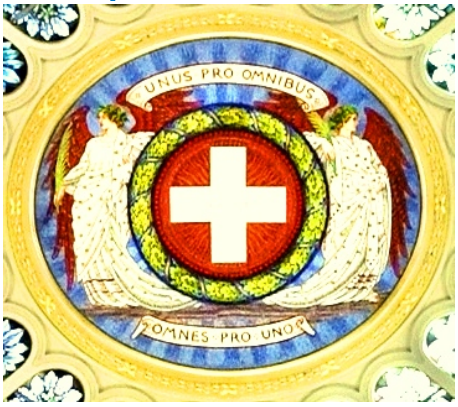

Unus pro omnibus, omnes pro uno

Mathematics cannot be vulgarized. Why?
Exactly due to chance, unexpectedness, due to the fact that she is not one.
No way to open some vast avenues which we can look through without entering it.
It is necessary to enter inside.
Simone Weil (1909-1943), Cahiers 1

Abstract

We discuss the many aspects and qualities of the number one: the different ways it can be represented, the different things it may represent. We discuss the ordinal and cardinal natures of the one, its algebraic behavior as a neutral element and finally its role as a truth-value in logic.

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[1] This paper has been written in such a way that it can be understood and/or tasted by any gentleman or gentlewoman with an average IQ but is not recommended for people with an emotional intelligence of less than $\aleph_1$.

This is a revised version taking in account the reflections of many ones including I, me, mine. I am grateful to Jonathan Westphal, Michele Friend and Srećko Kovač who carefully read the original version and indicated in particular (but no only) English incorrectnesses and/or topys (like this 1). Thanks also to Jasmin Blanchette, Catherine Chantilly, Newton da Costa, Luis Estrada-González, Yvon Gauthier, Paul Healey, Wilfrid Hodges, Jean-Louis Hudry, Colin James III, Jens Lemanski, Dominique Luzeaux, , Daniel Parrochia, Arnaud Plagnol, Andrei Rodin, Denis Saveliev, Matthias Schirn, Sergey Sudoplatov, José Veríssimo, Gereon Wolters and John Woods for their useful comments and/or feedback.

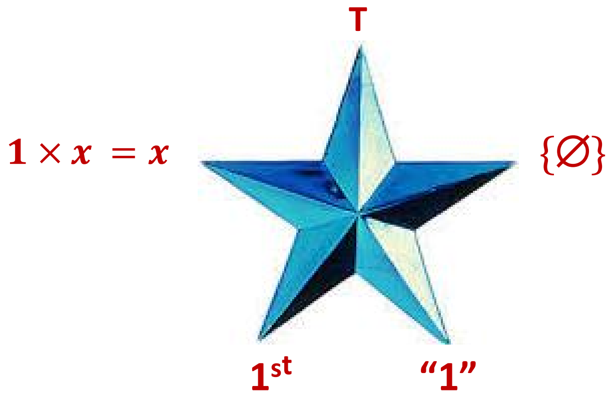
I would like to dedicate this paper to my mother who gave me birth, introduced me to modern mathematics through the work of Georges Papy (who passed away on 11.11.11) and provided me access to the pathless land.

Table of Contents

0. One Five-Pointed Star

In this paper, we distinguish five different main meanings of the number 1 showing that plurality is at the heart of one fundamental notion of mathematics. This is the opportunity to discuss many aspects of mathematics.

The number one is a good starting point for a trip into the mathematical universe; for us however it will not be just a starter but also the main course. Our present journey will stick around this one, yet it will not be a binding stake, rather a guiding compass, a star in the sky of thought. Our journey can be represented by the following five-pointed star:



After starting from the South Oriental position with the Digital One, we go 1^{st} horizontally on the West, then we transversally go up to a Cardinal Eastern Singleton. Going again West we horizontally reach One Neutral position, before raising the True One at the Top.

This journey is therefore not like a straightforward promenade along the avenue of *Champs-Elysées*, smoothly going down from the *Arc de Triomphe de l'Etoile* up to the Obelisk of Luxor in the *Place de la Concorde*, stopping to have a glass of champagne at the Fouquet's. Nor, despite its relative shortness (that's not one's monograph), is it similar to a supersonic trip from Paris to New York in 3h30 on board the Concord glancing over the Atlantic Ocean confusing icebergs with clouds, whales with cruise ships, the air hostess with a nurse.

Although our path is more like a zigzag, we will avoid being diverted by irrelevant details, which may pop up on the way, like goblins in the forest, or one of those things. We will go deeper and deeper, higher and higher, starting from the superficies of names and notations of the one and going to its true peak or kernel, if any. We hope, at the end, to reach concord between many ones, if not to be at one with at least one reader, without statufying one's liberty.

Along the way, we will meet many characters such as Al-Khwarizmi, the Empty Set, Sebastian Vettel, the Existential Quantifier, Nicolas Bourbaki, the Pirahã from Amazon, Zurich Axioms, Garrett Birkhoff, Descartes's Tree, Kangaroos, the Monopoly, David Hilbert, Boolean Algebra , the Hexagon of Opposition, Category Theory, Wacław Sierpiński and many more.

Let's take One for the Road …

1. One

1.1. Many Names, One Digit

One is not “one”. The name is not the thing, and there are many ways to name the number one. This is also true for other things. The plurality of names for one and the same thing is common. The number one can be called just “one” or more extensively “the number one”. The second name emphasizes the fact that this is a number, “one” having other meanings, not completely unrelated, as when used as an indefinite pronoun: “one would like to know what one is”. And despite these disparities, the number one itself is called “unity”.

Moreover, besides the variations of names in one given language, there is the multiplicity of languages. In nearly each natural language, the number one has a different name, even if there is sometimes a common root: *un*, *uno*, *unum*, *um*, etc. Following a linguist structuralist stream à la Saussure, one may argue that each of these names has a different meaning, although these different meanings have a common kernel. In an indigenous language, like the one of the Pirahã of Amazon (see Frank et al. 2008), where we have «one, two, many», the meaning of the word for one is not the same as the one for “one” in English.

A name for one works quite like a proper noun. A proper noun, by contrast with a common noun, designates a *singular* object – for example the “Eiffel Tower”, “Napoleon Bonaparte” – although the oneness of the object may not be so easy to single out, like in the case of “Paris”, “France” or “French”.[2] Proper names in English are capitalized and they vary less from one language to another than common names, sometimes not at all, like “Nicolas Bourbaki”, unless we change of alphabet; in Russian this leads to: Никола Бурбаки, and in Hindi to: नीकोला बूरबाकी . However, the name for one, although it singularly designates one thing, is generally not capitalized and varies a lot, like the name for the sun.[3] But there is another name for one which almost does not vary and that it is not absurd to consider as capitalized:

[2] Proper noun: the name of a particular person, place, or object that is spelled with a capital letter.
Common noun: a noun that is the name of a group of similar things, such as “table” or “book”, and not of a single person, place, or thing. (Online Cambridge Dictionary)

[3] There are other properties that challenge the classification of “one” as a proper noun: «Proper nouns are not normally preceded by an article or other limiting modifier, as any or some. Nor are they usually pluralized. But the language allows for exceptions» (Dictionary.com). One exception proves the rule …

This is also the case of the other nine digits: 0, 2, 3, 4, 5, 6, 7, 8, 9 and the infinite series of names they generate by combination. These digits and the mechanism to combine them, called "algorism" (*algorismus* in Latin), have an Indo-Persian origin. The word "algorism" is derived from the name of the famous Persian mathematician, al-Khwarizmi (780-850) (*Algoritmi* in Latin), who was a bridge between India and Occident (derived from him are also the words "algebra" and "algorithm"). About these digits, Karl Menninger (1898-1963) wrote in his book, *Zahlwort und Ziffer - Eine Kulturgeschichte der Zahl (Number words and number symbols – A cultural history of numbers)* in 1934[4]: "These ten symbols which today all people use to record numbers, symbolize the world-wide victory of an idea. There are few things on earth that are universal, and the universal customs which man has successfully established are fewer still. But this one boast he can make: the new Indian numerals are indeed universal" (Menninger 1970, p.391).

And among the ten digits, "1" is probably the most famous, although the digit for zero "0" is nearly overshadowing it. The weak point of "0", however, is that it is not clearly and immediately identifiable. It can be confused with the glyph "O" for the 15th letter of the alphabet or with a circle. In other words, it does not necessarily appear as a numeral or even as a sign. It is true that in some fonts the glyph for one is also quite similar to the glyph for a letter of the alphabet, for example in Times New Roman, we have "1" and "l"; by contrast in this font we have a better distinction between "O" (letter) and "0" (number).[5] Then the sign for zero looks much more like an oval and the glyph for the 15th letter much more than a circle. Funnily enough, this letter in English is the initial letter for the name for the number one, which in several cultures has been represented by a circle. Even if this circle may sometimes just looks like a fattened dot: ●(Mayan numeral for one), it is interesting to note the relation between this geometrical shape and the number one; this can be interpreted in different ways: mathematically (a point[6], a circle) or physically (an atom).

How to symbolize any one single thing? It has to be anonymous: it can be a small circle, a small line (horizontal or vertical), a small cross that will characterize this ontological deflation. Louis-Gustave Du Pasquier (1876-1957)[7] writes: "What do we retain, when we operate the abstraction necessary for counting (*dénombrement*)? Each perception is contemplated as something, anything, as an entity, as an *ens*, as a (*un*) 1" (Du Pasquier 1921, p.38; in parentheses the original French words). We may wonder if such *dépouillement*[8] of reality is the exclusivity and/or privilege of mathematics. We don't necessarily need to count or

[4] Many books have been written about the story of numbers, this is a classical one. Other interesting ones are (Dantzig, 1930), (Ifrah, 1981) and more recently (Corry, 2015). The literal translation of the original French title of Ifrah's book would be *The universal story of numbers* with subtitle *The intelligence of human beings narrated through numbers and counting.* It has been creatively translated in English as *From One to Zero: A Universal History of Numbers*. This book is an interesting best seller but weak points have been pointed out, especially regarding the last part about the development of computation, see e.g. Dauben's critcisms (Dauben, 2002). Joseph Dauben (1944-) is the editor of an annotated bibliography about the history of mathematics (Dauben 1985) where the reader will find indications about many other books concerning numbers.

[5] To avoid confusion the glyph for the number zero is sometimes slashed, but this convention has been used the other way round by IBM and in mathematics this is similar to the empty set; the glyph for number 1 has not yet been slashed. The capital letter "I" can also be confused with symbols for the number one. "I" incarnates the first person, not any one.

[6] From this point of view, one is the starting *point* of Geometry.

[7] Du Pasquier, a former student of ETH Zurich, was professor at the University of Neuchâtel and responsible for the edition of part of Leonhard Euler's *Opera Omnia.*

[8] This is an untranslatable French word, which, punnily enough, has a double meaning: the primary one meaning *taking out*, the secondary one meaning *counting the votes*.

enumerate to have the idea of an indeterminate thing, and in some sense 1 is a notion primary to counting and enumerating, as it appears both in the dichotomy one/many and in its one-to-one aspect that we will examine in section 1.3.

The horizontal line "—" is a glyph that has been chosen for the number one in various cultures, but the vertical one won. It is not imperative to develop an explicit analysis of the sexual nature of this phenomenon; a picture is worth a thousand words. The one below, anticipating the first meaning of 1 discussed in the next section, is a photo of Sebastian Vettel (1987-) commemorating on a podium one of his many number one positions at a Formula 1 Grand Prix:[9]

Combinations of horizontal lines have been used in the *I Ching* to produce a kind of universal language with 64 hexagrams. Gottfried Leibniz (1646-1716) was quite fascinated by such a construction (cf. Perkins 2004), he who had the idea to develop a mechanical system for thought, anticipating computation,[10] which, funnily enough, is using sequences of 0s and 1s, as a universal language, based on "bit". But this is rather a codification than a language (see Mackenzie, 1980). Two digits are enough to codify, but not necessarily to think. As René Guitart (1947-) puts it: "nothing that is real can be codified" (Guitart, 2000, p.42).

Other famous mathematical symbols are the symbol for equality "=", introduced in 1557 by the Welsh mathematician Robert Recorde (1512-1558)[11] and the symbol for square root "√", introduced in 1525 by Christoff Rudolff (1499-1545) and finalized in 1637 by René Descartes (1596-1650) adding the vinculum. Mathematical language is universal, in the sense that all mathematicians in the world use quite the same notation. The equation:

$$1 = 1$$

[9] The phallic aspect of the "1" contrasts with the feminine aspect of the "0". Both are present in mathematics, but like Adam, 1 was the first. They can be seen as the two main characters of the story of mathematics, from which everything is springing.

[10] This part of the work of Leibniz was mainly rediscovered by Couturat (1901). In this seminal book he gives a good analysis both of *lingua universalis* and *calculus raciocinator*. Couturat is also the author of a comprehensive book of 700 pages on the history of the universal language (joint work with Leau in 1903) and an interesting paper on the logical definition of number (Couturat 1900). The correspondence between Couturat and Russell which was frozen during many years in La Chaux-de-Fonds, Switzerland, was published in 2001 by Anne-Françoise Schmid.

[11] As we have pointed out in a paper on identity (Beziau, 2015), Recorde's symbol is, like the balance of justice, a double symbolization: a pictogram representing an object which is a prototypical specimen of the thing it signifies. About the process of symbolization see our book *La pointure du symbole* (2014), related to a congress we have organized at the University of Neuchâtel in 2005 on symbolic thinking, including two papers on mathematical symbolism : (Robert, 2014), (Pont, 2014).

is written in the same way in any country and can be understood by anyone. Mathematics in general is the most unifying thread in human culture. As David Hilbert (1862-1943) put it: “Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country” (quoted in Eves, 1971). At the end of the 19th century many people tried to create a universal language for humanity but they didn’t succeed to implement one, besides the mathematical one. Among the various proposals was *Esperanto*.[12] In Esperanto the name for one is “unu”. Esperanto, similarly to *Interlingua* promoted by the Italian logico-mathematician Giuseppe Peano (1858-1932), is mainly rooted in Latin.

Mathematical language has been developed under various influences. The way it works is more similar to a pictogramatic language like Chinese, than an alphabetic language. In particular, what predominates is the use of a single sign to denote one thing. This is typically the case of “1”. By contrast, an alphabetic language use several signs, a *word*, to denote one single object. “One” has three letters (but “I” is a one letter word). This situation is paradoxical if we think that on the one hand the prototypical example of alphabetical language is the Greek language, and that on the other hand mathematics is supposed to be born in Greece, considering that proof is the quintessence of mathematics.[13] This shows that the notion of proof does not depend so much on notation.[14]

The number one has a universal name, namely “1”. On the other hand, “1” is not really a name for only one object, but for a plurality of objects: one as a natural number, one as an integer, one as a rational number, one as a real number, etc. All of them are denoted by the same sign: “1”. We can say that these numbers are different because they have different properties. But the reason that the same sign is used is that they also have something in common. In particular, they all obey the following axiom:

$$1 \times x = x$$

In Model Theory, this would instead be written as

$$\forall x \quad \mathbb{1} \times x = x$$

where “$\mathbb{1}$” is a sign that can be interpreted in different ways. This idea was implicit in the “symbolic algebra” of George Peacock (1791-1858) (see Durand-Richard 2007 and Grattan-Guiness 2000) and was explicitly systematized only 100 years later by Alfred Tarski (1901-1983) (see Tarski 1954-55 and Chang/Keisler 1973).

One is also called “unit” corresponding to the fact that 1 is used as a unit for a variety of measures ranging from weight to money through space and time. The logic of this variation is not always clear both from within and from the outside. 1 km is not the same distance as 1 mile and what is the relation between 1 km and 1 kilogram? Of course it is always possible to establish some superficial connections: running 1 km in 1 min one may lose 1 kilo. And as we know time is money: in 1 hour one may earn 1 dollar. However, despite the book by Max Gunther (1927-1998), with the suggestive title *Zurich Axioms* (1985), it is not clear that economics can fully be axiomatized.[15] Is there a categorical axiomatization for economics in 1st order logic (classical or not) explaining the relation of the following money units?

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[12] Practiced by Rudolf Carnap (1891-1970), a member of the abortive project of a unified science including an *International Encyclopedia* ; see (Carnap, 1934), (Morris, 1960), (Salmi, 2012) and *Otto Neurath and the unity of science* by Symons et al. (2011) published in the book series *Logic, Epistemology and the Unity of Sciences* (Springer, Dordrecht) launched by Shahid Rahman as a way to take up the torch.

[13] “Qui dit mathématiques, dit démonstration”; this is the famous opening of *The Element of Mathematics* by the General Bourbaki (1934-). This has been reinforced by one of his disciples, Jean Dieudonné (1906-1992).

[14] See the recent work by (Guo 2014) for a comparative study of Greek rationality and Chinese thinking.

[15] See e.g. the paper by Tsuji, da Costa and Doria (1998) stating in particular negative results about Nash equilibrium.

Zurich bankers were using axioms in a successful, but rather metaphorical way (Fritz Zorn should not be confused with Max Zorn).

Bourbaki has emphasized that the axiomatic method is a way to unify mathematics; this has been pointed out in his famous piece *The Architecture of Mathematics* (1948):

> Where the superficial observer sees only two, or several, quite distinct theories, lending one another *unexpected support* through the intervention of a mathematician of genius, the axiomatic method teaches us to look for the deep lying reasons for such a discovery, to find the common ideas of these theories, buried under the accumulation of details properly belonging to each of them, to bring these ideas forward and to put them in their proper light.

However, this modern axiomatic method is itself relative, as Bourbaki wrote: "It goes without saying that there is no longer any connection between this interpretation of the word *axiom* and its traditional meaning of *evident truth*"; and talking about the choice of axioms: "There is nothing absolute in this choice; several systems of axioms are known which are *equivalent* to the one which we are stating explicitly, the axioms of each of these systems being logical consequences of the axioms of any other one."

The introduction of this piece is entitled *La mathématique ou les mathématiques*? (Literally: "*Mathematic or mathematics*?"). The name Bourbaki was coined by André Weil when in India. Weil was an aficionado of Indian culture. He read the *Bhagavad Gita* in Sanskrit (see Weil 1991). "Nicolas Bourbaki" is a proper name, like "Napoleon Bonaparte", it is supposed to denote one single character. But it was used by André Weil and his friends to denote a group of mathematicians who promoted modern mathematics. The proper name Bourbaki gathers a multiplicity into a unity, and the members of Bourbaki were behaving as a unified community, releasing their books only upon unanimous approval (see Beaulieu, 2006). Their behaviour was in harmony with the search for unity in mathematics, replacing the plural "les mathématiques" by the singular "la mathématique". The title of their main work is Eléments *de Mathématique* (*Elements of Mathematic*).

In French, the plural of "mathématiques" is rather suspicious. In English, despite the "s", "mathematics" is singular (and it was syntactically singular up to the 17th century) , like "physics", "politics", "linguistics", etc. In most languages, it is singular and feminine when there is a gender disparity, as in Portuguese (*a matemática*), even in languages where there is neutrality, like in German, "Die Mathematik" is feminine (but the original Greek is indeed neutral). The French plural is feminine and Bourbaki uses the feminine singular. The General did not invent it but gave a new meaning to it. "La mathématique" was already used in French, and still is in use, to talk about the more philosophical essence of mathematics.

Since in German the feminine is used by the working mathematicians, Martin Heidegger (1889-1976) in *Die Frage nach dem Ding* (1935-36) (*What is a thing?)* introduced a masculine version, "Der Mathematik", for the quintessence of mathematics. In this book, Heidegger points out the close connection between mathematics and reality. He emphasizes the etymology of the Greek word τα μαθήματα (*tà mathémata*): *Things as they can be learnt*. From this point of view, the mathematical is the key to reality, a key that can be interpreted either in a Pythagorean way or a Kantian way (Heidegger's book is about Kant's philosophy, the subtitle is *Zur Kants Lehre von den transzendentalen Grundsätzen*).

1.2. The First One

A good quality of the number one is to be the first. This has to do with its ordinal nature. In ancient Greece, the sign used for the number one was "α", because it is the first letter of the Greek alphabet. The *alphabetic order* is a typical example of total discrete order with a first element. The order of the alphabet is arbitrary in the sense that there is no *reason* why "α" is the first, "β" the second and so on. Can we say that there is a good reason for 1 being the first?

There is a simple and trivial answer to this question: to be the first is the very nature of the 1. But this has to be relativized: to be the first is the very nature of 1 as an ordinal number, of 1 as the first. At the end, we have a true tautology: the first one is the first.

What is interesting is to explore the relation of this quality of 1 with its other qualities. This touches on a central question: is there a real unity beyond the multiple aspects of 1? Maybe not.

And there is a more dramatic feature: 1 has lost its firstness. In Greece, 1 was really the first because there was no 0. If nowadays we consider the natural numbers, they form a total discrete order with a first element which is 0, not 1. 0 is the first.

The existence of a first element of an order relation is given by the following axiom

$$\exists x \forall y \quad x \leq y$$

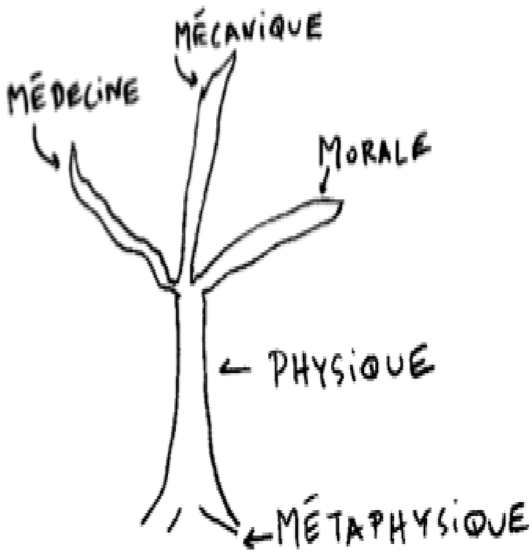
It is true that an object a such that

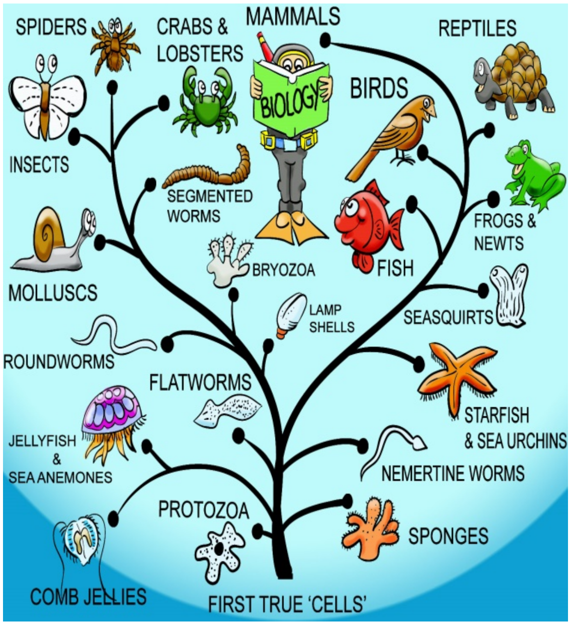
$$\forall y \quad a \leq y$$

is called a *first* element and firstness is associated with 1, it is written "1st". But it can be another number than 1. This in fact points to the disparity of the qualities of 1, in particular between ordinality and cardinality. In set theory, the empty set is the first, it obeys the axiom:

$$\forall y \quad \varnothing \subseteq y$$

The relation of inclusion denoted by "$\subseteq$" is a partial order. The notion of first element can be generalized to partial orders, which in some cases can be represented as trees where the root is the starting point.

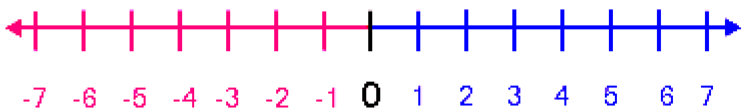




In set theory, 1 arrives in the second place, it is the set with the empty set as only one element: $\{\varnothing\}$. To represent the cardinality one, there must be something before one. One cannot be the first ordinal.

Nevertheless 1 keeps a central position in the structure of natural numbers, because it is the difference between each number. The *succession* of numbers is based on one. 1 can be qualified as the "succession unit". Commenting on Peano, Paul Halmos (1916-2006) writes: "the popular proposition *two and two make four* can be written in the unabbreviated form: (1 + 1) + (1 + 1) = 1 + (1 + (1 +1))" (Halmos, 1977, p.6) and Arnaud Denjoy (1884-1974) makes the following comments: "The number of the simplest species is the positive integer. Positive integers are ordered. Each of them, by definition, is obtained by adding a unity to the one which precedes him. Thus it is enough to define 1 and the addition of 1 to a supposedly known positive integer, so that 2, 3, ... are defined one after the other: the integer n being defined, the addition of 1 defines $n + 1$. We admit that nothing allows conceiving that this construction stops. It is what we express by saying that the sequence of positive integers is *indefinite*. It is the first appearance of the infinity in mathematical." (Denjoy, 1937, 1-68-3). In the next section we will see another connection between 1 and ∞.

This feature of succession unit is also kept in the structure of the integers, where 1 is duplicated, we have +1 and -1, like all natural numbers except 0.[16] We can say that +1 is the first positive number and -1 is the first negative number. On the other hand strictly speaking the structure of the integers $\mathbb{Z}$ is a structure without a first element:

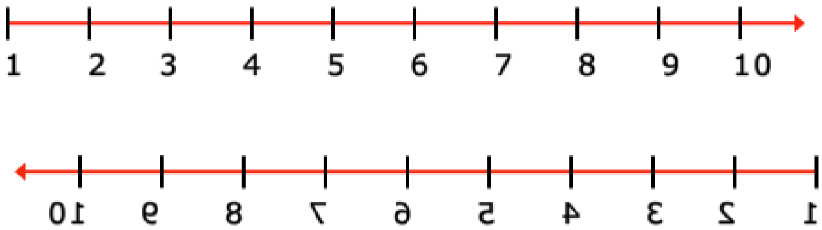


But $\mathbb{Z}$ can be well-ordered, i.e. organized in a way such that every subset of it has a firstelement. This is also the case with the rational numbers, in particular via the standard diagonalization showing that $\mathbb{Q}$ is denumerable. By contrast diagonalization was also famously used by Georg Cantor (1845-1918) to show that the set of reals is not denumerable, that there

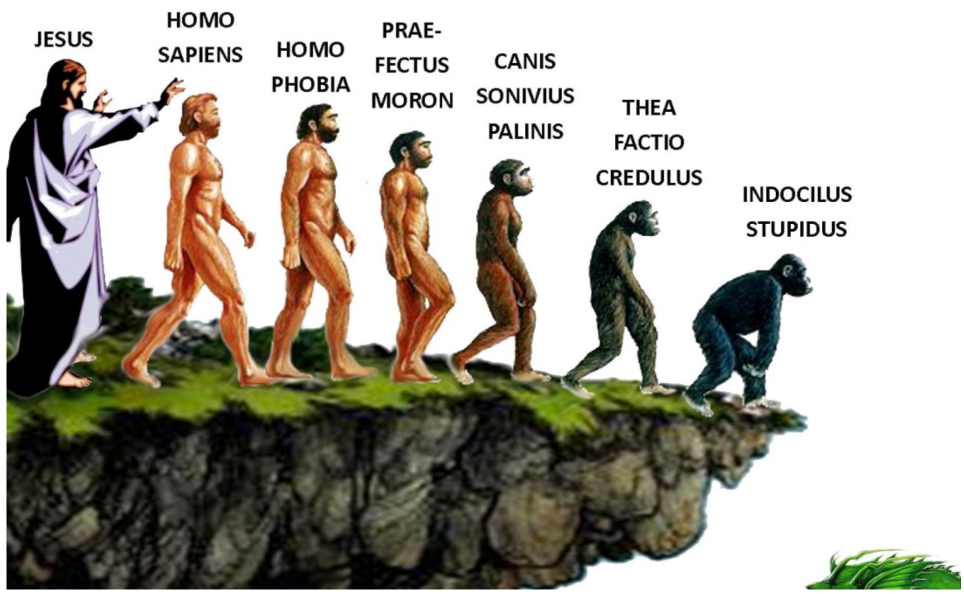
[16] It is also possible to attribute a sign to zero, cf. the IEEE 754 for floating-point arithmetic.

is not only one infinity. The set $\mathbb{R}$ can be well-ordered only using the axiom of choice. Wacław Sierpiński (1882-1969) proved in 1947 that Zermelo-Fraenkel set theory + GCH (the Generalized Continuum Hypothesis stating that the cardinality $2^{\aleph_0}$ of $\mathbb{R}$ is the first one after $\aleph_0$, the one of $\mathbb{N}$, thus named $\aleph_1$, and so forth) implies the axiom of choice (this was conjectured by Lindenbaum and Tarski in 1926), hence that $\mathbb{R}$ can be well-ordered. In other words: if the cardinality of $\mathbb{R}$ is the first one after the cardinality of $\mathbb{N}$ and so on ($2^{\aleph_1}$ is the next cardinal after $\aleph_1$, etc.) then $\mathbb{R}$ is well-orderable.

But in any case, we can consider a segment of the real numbers $\mathbb{R}$ having both a first and last element, like the interval [0,1]. 0 is the first, 1 is the last, but this seems quite arbitrary. It is also quite arbitrary to consider 1 to be the first in the set of positive integers. The relation of order is admittedly antisymmetric, but it is reversible!



As written in the *Gospel of St Matthew* (20:16): “So the last shall be first, and the first last.”

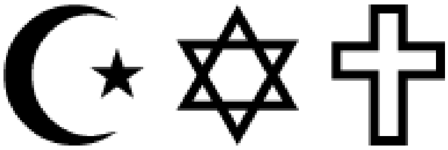


Contrasting with this *ordinal* position of exception, the first or the last, is the *ordinary* one. The ordinary one is more explicit in Latin languages: *Tem um gato na praça* (there is one/a cat in the square). In Portuguese “one” corresponds to the indefinite “um” article by contrast to the definite article “o” (In English there is also etymologically a connection between “an” and “one”).

O gato esta na praça means *The cat is in the square*. *Aquele gato*: the famous cat, the star. *O*ddly enough we have here again O versus 1, O having the winning strategy. The “o” is singled out, the “um”, is no *one*.

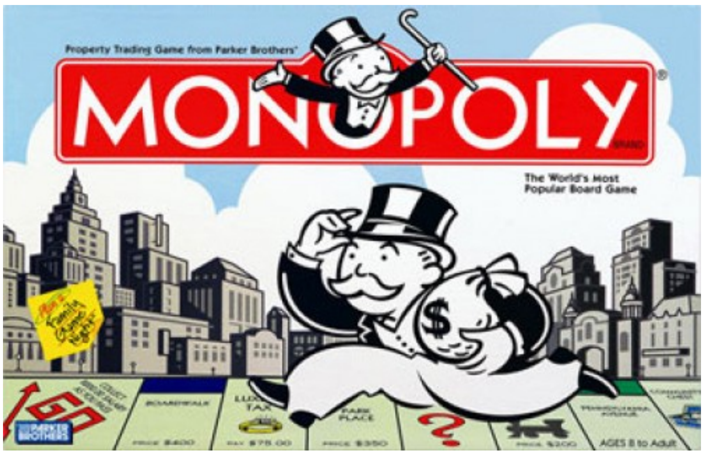
1.3. The Only One

The quality of 1 as a cardinal number can be qualified as “uniqueness” or “unicity”, meaning there is only 1. It manifests in natural language through the Greek prefix “mono”: “monotheism” means a religion with only one God, “monogamy” means a relation with only one woman, “monograph” a book with only one subject, etc. There is a plurality of monotheist religions. The three main ones are Judaism, Christianism and Islam. Despite their monastic nature, they don’t necessarily agree on everything. For example, polygamy is authorized in Islam, not in Christianity or Judaism; nevertheless if we consider the respective sacred books, *The Koran*, *The Bible*, *The Torah*, each one of them is a monograph.



Oneness is a singular cardinality in the sense that we have the opposition between *one* and *many*, a classical dichotomy, part of Pythagoras’ table of opposites. This dichotomy has been quite fertile, generating many offspring, notably Plato’s *Parmenides*, and more recently *L’être et l’événement* (1988) by A.Badiou (1937-) and *One* (2014) by G.Priest (1948-).

Contrarily to the appearance, the famous board game “Monopoly” is not one of these offsprings. “Poly” is not here derived from the Greek word meaning many, as used in “polygon” or “polysemy”, but from πωλεῖν (*pōleîn*, “to sell”). This game was originated by Elizabeth Magie (1866-1948) who was an American anti-monopolist. Monopoly is a famous plague. *Microsoft* was a couple of years ago fined the equivalent of US 2.5 billion by the European Commission for this disease (*New York Times*, Feb 28, 2008).

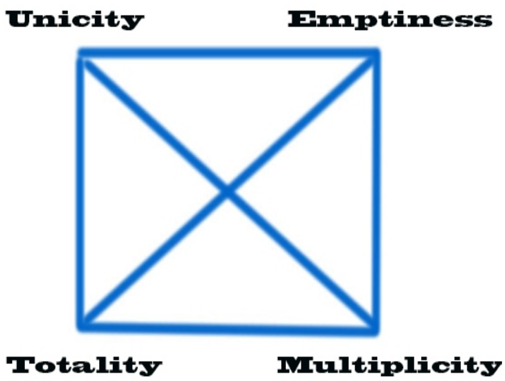


We are talking here about the business world but monopoly is certainly also not so good in the intellectual world. It can be manifested in different ways: the monopoly of a science over the other ones, like physics (leading to physicalism), and in a given field through the idea of *main stream*, oscillating from a conservative trend to a fashion or a hybrid combination of both, in any case pretty gregarious movements.

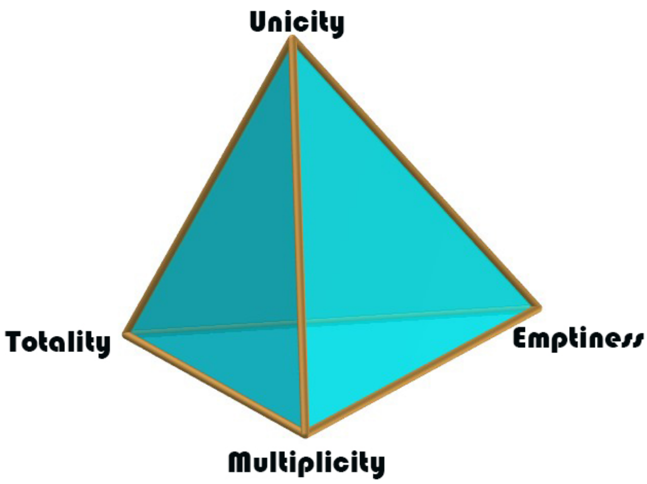
Mathematics has in some sense monopolized modern sciences; although it is not clear to what extent it is the case in the human sciences. Mathematics has itself during centuries been monopolized by quantities, magnitudes, measurements. It was set free from this bondage in particular by Boole, the initiator of modern logic, developing algebra with non-quantitative objects (see our recent paper on Boole). After Boole, there was a new wave of monopoly with set theory. And there are still some people who want to build a big monolithic grounding theory with arrows or whatever.

Too much reduction is not good but proliferation also leads to chaos. What is important is to find a good equilibrium. Dichotomy is the basis of plurality, and in some sense it is enough: with 0 and 1 we can generate all the natural numbers and also all the real numbers. But an alphabet of about 25 letters is more appropriate for a human language. The duality one-many can be split into a (trans) infinity of cardinal numbers. However, it also makes sense to reduce this multiplicity to four cardinal cardinalities.

We can have an extended picture of the situation with four basic notions: emptiness, totality, uniqueness and multiplicity (excluding totality, i.e. many but not all). This forms a square of contrariety: [17]

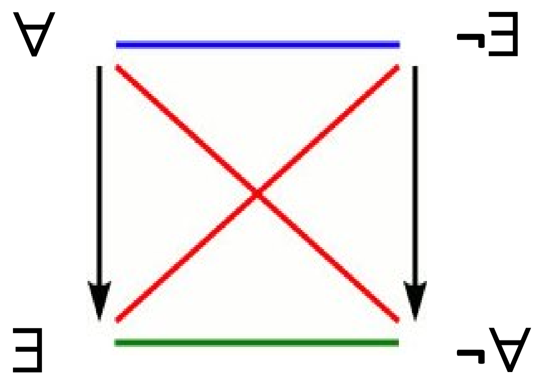


Following the idea of A.Moretti (2009) a square of contrariety can better be represented as a tetrahedron (where there is the same distance between each pair of vertices). We therefore have:

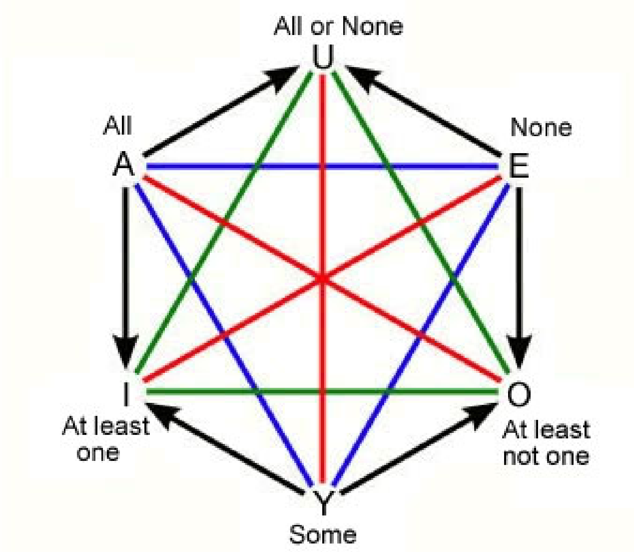


[17] For the definition of contrariety see for example our recent paper "Disentangling contradiction from contrariety via incompatibility" (2016), and for other work on the square of opposition, see in the bibliography of the present paper the volumes we have edited on the topic.

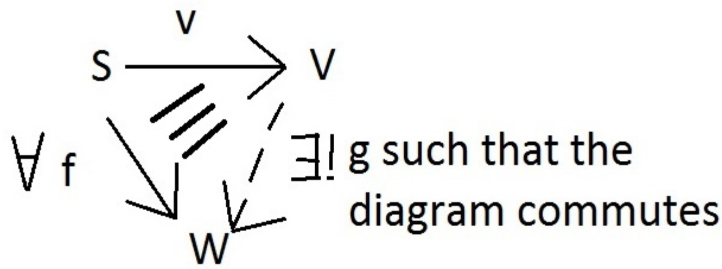
In the theory of quantification, generally only two quantifiers are considered as primitive: the universal quantifier and the existential quantifier.[18] But we can easily construct a square of opposition with four quantifiers:



And the following hexagon, put forward by Blanché, gives a more precise view on quantification (see our 2012 paper, “The power of the hexagon”), and frees quantification from existentialist problems (see our 2005 CQFD paper) :



The dominating quantifier is generally “at least one”, not “one and only one”, which is represented by “∃ !”. However, existence and uniqueness is a leitmotiv in category theory:

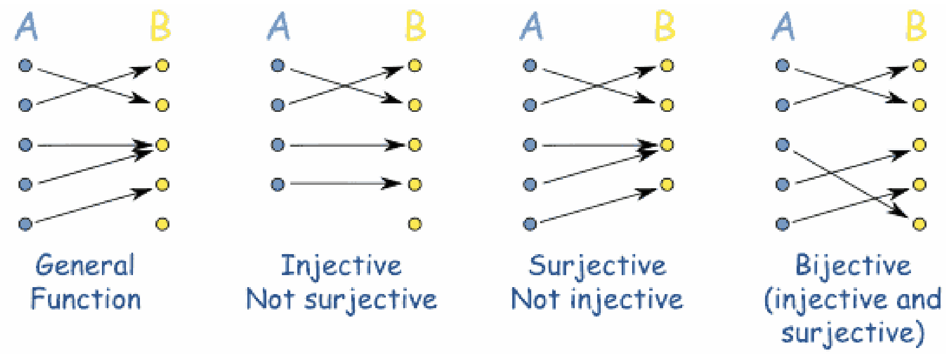


[18] Both ∀ and ∃ are generally used, although it is well known that, in classical logic, it is enough to take one of them as primitive.

The role of 1 in cardinality theory does not reduce to uniqueness. There is the notion of one-to-one correspondence (put forward by Cantor). Two sets have the same cardinality iff there is a one-to-one correspondence between them. The identity or difference between cardinalities is not based on numbering in the sense that we don't have to count: if we see a room with people seated in every chair, we know that there is the same number of people as the number of chairs, whatever the number is.[19] And the idea of one-to-one correspondence is a key to infinity: a set A is infinite iff there is a one-to-one correspondence between its elements and a proper subset of A (Dedekind's definition). This is a link between 1 and all the infinites.

Many sets have 1 as cardinality: $\{\varnothing\}$, $\{\{\varnothing\}\}$, $\{\{\{\varnothing\}\}\}$, … . On the other hand, any set can be seen as a unity even if its cardinality is many. But in the standard ZF set-theory, a collection of sets forms a set iff there is a set s such that each member k of this collection is a member of s, i.e. $k \in s$. The collection of all sets in ZF, the universe, is not a set, as it cannot be unified. This is a way to avoid Russell's paradox. Cantor had already had the idea of the distinction between *consistent* and *inconsistent* multiplicities. Paradoxically, for Cantor a consistent multiplicity is a multiplicity which is one (set) (letter of Cantor to Dedekind, July 28, 1899).[20]

The concept of function is itself based on uniqueness. Every element has *one and only one* image. The concept of function is a fundamental tool of modern mathematics and logic. It is a universal concept which unifies mathematics, and it is based on uniqueness.



Let us remember that the family "injection", "surjection", "bijection" (with the corresponding adjectives "injective", "surjective", "bijective") was introduced by Bourbaki (MacLane and Eilenberg are respectively credited for "injection" and "injective, see Miller).

[19] This has been emphasized in particular in Section 4 of Chapter 1 of Dantzig's 1930 classical book on number.

[20] The remarkable correspondence between Cantor and Dedekind has been published by Emmy Noether and Jean Cavaillès in 1937 (see the reference in the name Cantor in our Bibliography).

1.4. From Monoids to Imaginary Units

As we have seen one typical proprerty of 1 is:

$$1 \times x = x$$

This applies to 1 itself

$$1 \times 1 = 1$$

This can be called an *algebraic* property of 1. 1 is called a *neutral* element, or an *identity* element, for the operation $\times$.[21] In this sense, 1 can be identified with identity.

This property can be generalized. In the case of a Boolean algebra of sets, the neutral element is the universe **U**, i.e. the set containing all the elements, and the neutrality is with respect to intersection. For every set A we have:

$$\mathbf{U} \cap A = A$$

The whole intersecting with the part is the part. This dis-embracing view of the relation between the whole and its parts is not necessarily satisfactory from a biological or political point of view. Can we say that the intersection of a mother with her child is the child? Up to a certain point ….

Here again 1 is not alone, 0 is close by. Zero is a neutral element with respect to addition:

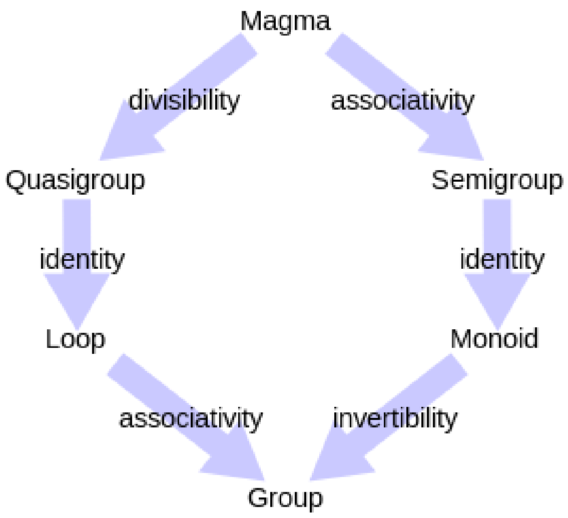
$$0 + x = x$$

A parallel situation is the one of the empty set, which is a neutral element with respect to disjunction in an algebra of sets:

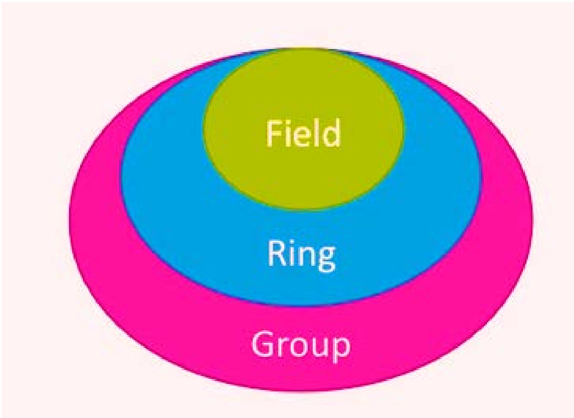
$$\varnothing \cup A = A$$

An algebra having a neutral element is called a *monoid*. An algebra with a binary function without any special property is called a *magma*. Monoids are structures between magmas and groups. A monoid is an associative magma – i.e. a *semigroup* – with a neutral element; its dual is a *loop*:

[21] Since we are taking examples of commutative operations, we are not making here the distinction between left and right identities.



After that, we have two other famous families of algebraic structures: ring and field.



The field of real numbers is extended to the structure of complex numbers in order to accommodate the imaginary numbers. The most illustrious imaginary number is the *imaginary unit*:

$$i = \sqrt{-1}$$

In the same way that square root of 1 has two roots, square root of minus one has also two roots, so there are two imaginary units: *i* ad –*i*.[22]

The general theory of algebraic structures is called *universal algebra*. The expression was coined by Sylvester in 1884, Whitehead wrote a large volume with this title in 1898 and the theory was furthermore developed by Garrett Birkhoff (1911-1996). He defined an abstract algebra as a set with a family of functions (or operators) without specifying any axiom. Birkhoff reached the stage of *axiomatic emptiness*.[23] As he explained in 1987, it was not possible to find common axioms for all known algebraic structures. There were in particular some incompatibilities between on the one hand the algebraic structures of the Noetherian School and on the other hand, the Boolean ones (including lattices, in which Birkhoff became one of the great experts).

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[22] The expression "imaginary number" is due to Descartes, the sign "i " to Euler; another connection between the number 1 and the 9th letter of the alphabet. About this use of the word "imagination" see our recent paper "Possibility, Imagination and Conception" (2016).

[23] About this notion, see our 2010 paper : "What is a logic ? - Towards axiomatic emptiness".

"Universal" is connected with oneness through unification. In the case of "universal algebra", unification is not axiomatic but rather conceptual.

The expression "Conceptual mathematics" is the title of a book by Lawvere and Schanuel with subtitle *A first introduction to categories* (1991). Category theory is more conceptual than axiomatical despite the fact it also uses some axioms (see our 2002 paper). [24] One of them is identity, also identified with one, it is a neutral element in a "functorial algebra", of which a *trivial automorphism* is a prototypical example (*triviality* is also a way to qualify the algebraic 1).

$$\begin{tikzcd}
A \arrow[loop left, "1_A"] \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[loop right, "1_B"] \arrow[d, "g"] \\
& C \arrow[loop below, "1_C"]
\end{tikzcd}$$

If we consider the category of all algebras of the same type, there is an *initial object* (a *first* one), an object such that there is a morphism from this object to all the other objects of the category. This is an *absolutely free algebra*, which is the basic structure of the syntax of propositional logics. Endomorphisms of this absolutely free algebra are substitutions. These tools were used to unify the study of a large family of propositional logics. The structure of their languages is the same. This general setting for logical systems was initiated in Poland by Tarski, Lindenbaum, Łoś and Suszko and was later on fundamental for the theory of combination of logics (see the book *Universal Logic: an Anthology*, 2012).

[24] About the development of category theory, see Mac Lane "Concepts and categories in perspective" (1989). Saunders Mac Lane (1909-2005) studied in Göttingen. "Conceptual mathematics" comes from Germany as explained by Sinaceur in her 1991 book *Corps et modèles* (pp.191-196).

1.5. The True One

In modern logic 1 is connected with truth. Mathematics is often considered as a paradigm of truth. As Mac Lane puts it: "philosophers' search for truth often will use the truths of Mathematics as the prime example of *absolute* truth" (1986, p.4). However, this is most of the time based on some misconceptions of mathematics such as considering that 2+2=4 is an absolute truth. Modern logic has seriously challenged this kind of absolute truth. Nowadays 2+2=4 is considered to be a relative truth, it is not necessarily true, some hypotheses or axioms are required, from which it is derived. Curiously this perspective that challenges mathematical truth is connected with the mathematization of reasoning where 1 appears as a central character, representing the truth-value called "true" on the basis of which absolute truth is defined.

The notion of truth or the word "truth" (or "true") barely appears in mathematical writings. At the end of a proof, what appears is Q.E.D., abbreviation of *Quod erat demonstrandum*, derived from the Greek, ὅπερ ἔδει δεῖξαι (*The very thing it was required to be shown*), and symbolized by Halmos by a black square "■", which in fact is at the same time the squaring of a fattened period, of a enriched full stop, meaning "end of the proof".

The notion of truth mostly appears at the metalevel: discussion about mathematics, the philosophy of mathematics. This metalevel is strongly connected nowadays with logic. For example in an *Encyclopedia of Mathematics* (see Weisstein, 1998 and Rehmann, 2002), related to "truth" we only find "truth-table" and "truth-value", basic notions of modern logic. And then 1 shows up, due to the fact that in modern times logic has been mathematized and 1 plays a central role.

In the so-called semantics of classical propositional logic (henceforth CPL), 1 is one of the two truth-values, called "truth" (or "true"), the other one being called "falsity" (or "false") and also represented by "0". Here zero is joining one more time one on the stage. Is "1" just a name for truth or is there a closer link between truth and 1?

The truth-table for conjunction is as follows:

∧	0	1
0	0	0
1	0	1

1 is a neutral element with respect to the truth-function of conjunction; we have:

$$1 \wedge x = x$$

Here we are using the same symbol "∧"for conjunction as a truth-function, as the one used for conjunction as a connective. Most people do that, but without consciousness. That is why an equation like $1\wedge0 = 0$ may look strange to them.

As shown by the above truth-table, x ranges over the two values 0 and 1. No quotes because we are not talking here of the names of these values, we could have said: x is ranging over the two values zero and one. But the numbers we are talking about here are not the standard zero and one. They are ones of them. We are talking of the zero and one of a Boolean algebra with two elements. According to this approach, truth is a mathematical object, which has a close connection with 1.[25]

Instead of using "1" as a name for the truth-value true, some people use "T", "T" being the first letter of "Truth". This has two disadvantages: on the one hand, the connection with

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[25] See details about that in our papers "Truth as a mathematical object" (2010), "A history of truth-values" (2012)", "Is the principle of contradiction a consequence of $x^2 = x$?" (2016).

one as an object is lost, and on the other hand this produces a confusion with the object $\top$, called "top" which is represented by a glyph very similar to the capitalized initial letter of the English word for truth.[26]

The terminology "top" and the sign "$\top$" are imported from lattice theory. It makes sense if we think of the connection between logic and algebra, in particular if we consider the Tarski-Lindenbaum algebra corresponding to CPL. But $\top$ can also be considered before algebraization. $\top$ is not a truth-value. From a syntactic point of view (the absolutely free algebra of formulas), it is a constant, which can also be considered as a 0-ary function, following a usual practice. From this point of view $\top$ is therefore a 0-ary connective. Considering CPL as a structural consequence relation, we have:

$$\vdash \top$$

In other words: $\top$ is always true. It is a *tautology*, a *logical truth*. In the factor structure of CPL, the Tarski-Lindenbaum algebra, all tautologies are reduced to one object, that can be symbolized by one of them, $\top$.[27]

The distinction between truth and logical truth is a fundamental distinction of modern logic. Is the number one closer to the truth-value 1 or to the 0-ary connective $\top$? It is in fact connected to both. $\top$ is also a neutral element in the sense that for any proposition p, we have (note that "$\wedge$" below is a connective not a truth-function):

$$\top \wedge p \dashv\vdash p$$

From left to right this is due to a basic property of conjunction (classical or non-classical). For any proposition q we have:

$$q \wedge p \vdash p$$

From right to left, if p is true, $\top \wedge p$ cannot be false, since $\top$ is always true and true plus true is true.

One could also use the name 1 for the object $\top$, this would not be more confusing than using the capital letter "T" for the truth-value true. In this case, we would write:

$$1 \wedge p \dashv\vdash p$$

Funnily enough, Garrett Birkhoff in his famous book *Lattice theory*, used "0" and "1" for bottom and top respectively, not "$\bot$" and "$\top$": "We shall use the symbols *0* and *1* to denote the (unique) least and greatest elements of a partially ordered system, whenever they exist. Thus in the system of all the subsets of any class, *1* denotes the whole class and *0* the empty set" (Birkhoff, 1940, pp.8-9). This is inherited from Boole, not because Boole was using "0" and "1" for truth-values, but because he was using them, respectively ,for the empty set and

[26] This second disadvantage does not manifest in Portuguese or in German where the first capitalized letter of the word for truth is not "$\top$" but "V" and "W" respectively. In English sometimes people use the lower case "t", in this case this second confusion is also avoided.

[27] The notion of tautology was essentially promoted by Wittgenstein (1889-1951), mainly in the *Tractatus*, but he did not reach the symbolization "T" for it and/or the corresponding top notion. Wittgenstein was against the Fregean notion represented by Frege (1848-1925) with "$\vdash$" (see Rombout, 2011). So the notation "$\vdash \top$" is completely anti-Wittgensteinian. Frege on the other hand did not make the difference between truth and logical truth, the symbol "$\vdash$" he introduced means for him truth, not logical truth, as used later on. So for him the notation "$\vdash \top$" would also be meaningless. Frege introduced in logic the expression "truth-value" and the corresponding two truth-values but did not represent them by 0 and 1, moreover although he considered them as objects, it was by contrast to the notion of function (see Heck and May, 2016). Furthermore we can distinguish in Frege the truth-value he calls "True" from truth which is not for him an object (see Greimann, 2007).

the universe, which are particular cases of least and greatest elements, when considering a lattice of sets.

The tautological 1 is in this sense more universal than 1 as a truth-value. The above axiom is valid for a wide family of logics, in principle for any conjunction whether truth-functional or not. The tautological one as symbolizing the notion of tautology is a universal feature of all logical systems. It is a unifying concept of the theory of logical systems — as is the concept of absolutely free algebra— more than one has a truth-value unless we consider non truth-functional semantics.

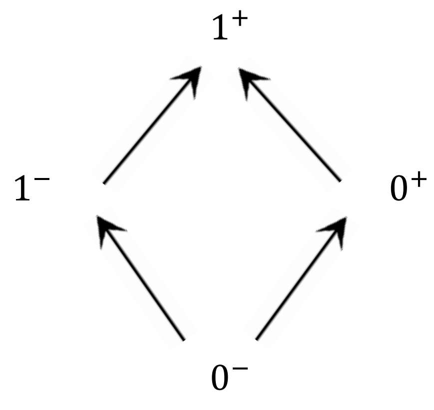
We can generalize two-valued matrix semantics to many-valued matrix semantics, for example to three-valued matrices. Then, we have for example the following table:

∧	0	½	1
0	0	0	0
½	0	½	½
1	0	½	1

This three-valued table corresponds to the one of Łukasiewicz's logic L3, where the third value ½ is not considered as designated. Later on, in paraconsistent logic, the third value was considered as designated, then ½ shares the truth with 1 . The above table leaves open the interpretation of the third value as designated or not.[28]

But generally, it makes more sense to use 0 and 1 with subscripts in many-valued logic, considering that the dichotomy between designated and non-designated values is indeed a dichotomy between truth and falsity. If we have a graded approach based on degrees, they should be interpreted on the one hand as degrees of truth, on the other hand as degrees of falsity (unless we go to graded consequence, cf. Chakraborty, 1988).

In a previous paper "Bivalent semantics for De Morgan logic (the uselessness of four-valuedness)" (2009) we have represented the four values of the four-valued semantics of Dunn-Belnap as follows:



1^+ and 1^- are the two distinguished values, both being two faces of truth.[29] As indicated by the title, in this paper we present a two-valued semantics for the corresponding logic, i.e. De Morgan logic. This two-valued semantics is non-truth functional.

[28] Initially Łukasiewicz (1920) used the symbol "2" for the third value, which is a bit absurd. He then shifted to ½. Asenjo, who was the first to develop paraconsistent logic based on three-valued truth-tables similar to the ones of Łukasiewicz, also used "2" for the third value, in paraconsistent logic it can make sense.

[29] E.Post (1921) originally denoted truth and falsity by "+" and "-". Our notation naturally allows the reading of this diagram as a bi-lattice taking in consideration this double reading.

Non-truth functional bivalent semantics were promoted by Roman Suszko (1919-1979) and Newton da Costa (1929-). The first one thought that proliferation of truth-values was absurd (cf. Suszko 1977), that objects of logical matrices are algebraic values, not logical values (a neo-Quinean position). The second one first developed this methodology to provide a semantics for his paraconsistent systems *Cn*, and then to develop a general theory of logics he called "theory of valuation" (cf. da Costa / Beziau 1994). In this theory, there are only two truth-values, falsity and truth and a tautology, as in classical logic, is defined as something which is always true.

There is the single 1, the truth value, and the universal 1, the tautology. But this universal tautological 1 is not unique, nor is it an absolute one. The notion of tautology is universal but its specification changes from one logic to another one.

These are tools that can be used to develop a theory of *universal logic* in a similar way to universal algebra (cf. Beziau 1994, 2006, 2012). From this perspective, universal logic is not *one* logic, *one* system of logic, it is not *a* universal logic, or *the* universal logic, in the same way that universal algebra is not *one* algebra, *an* algebra, *the* algebra. The unity is at another level. Universal logic is not a totalizing logic or a totalitarian logic but one general framework for the study of logics.

There is no one true logical system, because as we have recently pointed out in the paper "The relativity and universality of logic" (2015), on the one hand reasoning may vary according to the domain of applications, and on the other hand science is always evolving. Logic is relative but this does not exclude universality. Universal logic, as universal algebra, promotes unity in diversity. It is not a unity based on some fixed axioms or *principles*, like the principle of non-contradiction. It is not an exclusive unity, nor a static one, but a dynamical one, tracing a spiral.

2. One Conclusion

There are many ones, all having their own singularities. Each one has to be considered and respected. But it is much more interesting to develop cross-fertilization, mixing the fields, than pinnacling one one on one podium. Singledom leads to boredom and sterility.

It is fascinating for example to see on the one hand the way that 1 has transformed the theory of reasoning, being one key factor in its mathematization; on the other hand the way that this very theory of reasoning has given a new dimension to 1, according to which it makes sense to think that 1+1=1. That's Boolean algebra.

Unity is important in mathematics but there are different ways to proceed. Instead of favouring a big monolithic synthesis of the one or of the whole, it seems better to promote an *organic unity* to preserve the plasticity of mathematics.

This expression was used by Hilbert at the end of his famous Paris lecture (1900) and was taken up by Alain Connes (1947-) in his paper "A view of mathematics" (2009), starting with a first section entitled *The Unity of mathematics* where he writes:

> It might be tempting to view mathematics as the union of separate parts such as Geometry, Algebra, Analysis, Number theory, etc … This however does not do justice to one of the most essential feature of the mathematical world, namely that it is virtually impossible to isolate any of the above parts form the others without depriving them of their essence … the corpus of mathematics does resemble a biological entity which can only survive as a whole and would perish if separated into disjoint pieces.

When considering mathematics as an organism,augmenter it makes sense to conceive of it as a complex organism with a head and it seems natural to identify this head with logic (cf. Hilbert, 1923, relating absolute truths with proof theory, he also called *Metamathematik*). From this perspective, one mathematician, when rejecting logic, is cutting off one's head.

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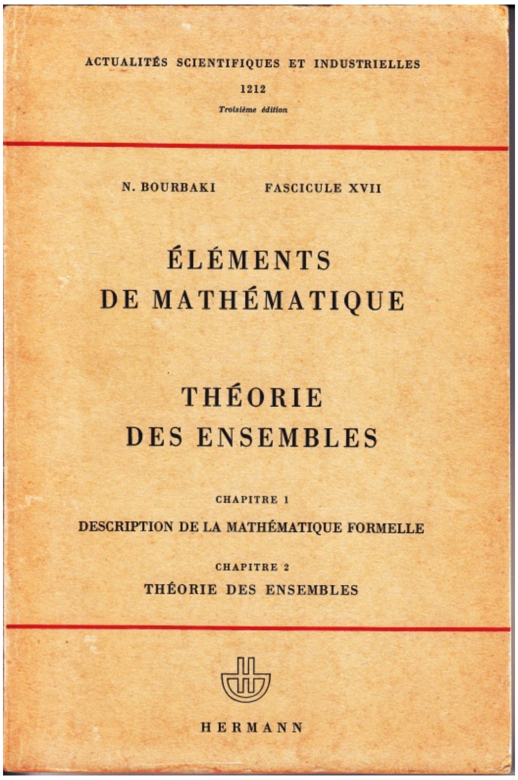

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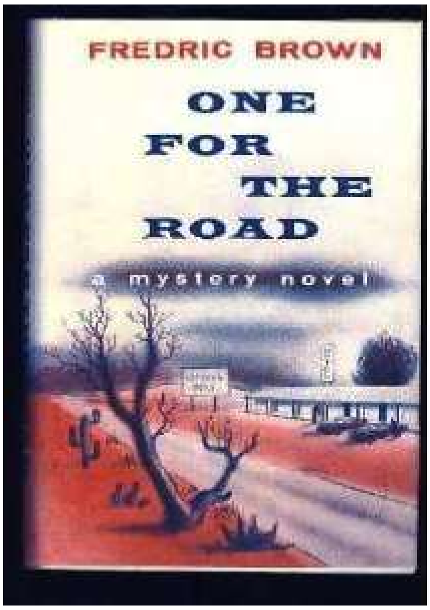

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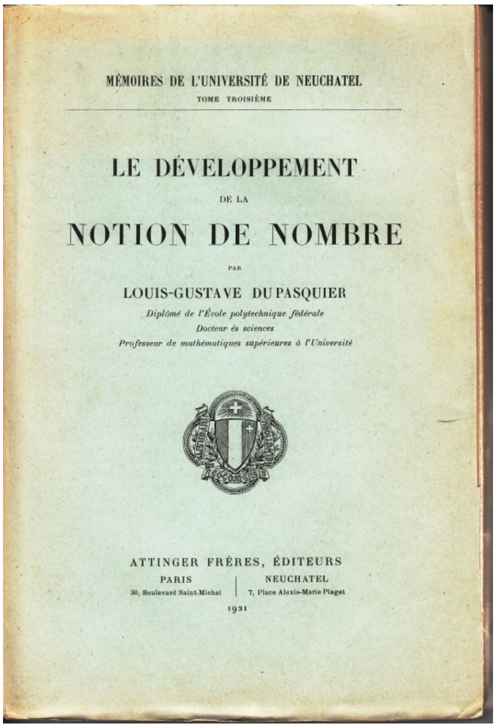

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3.2. Acknowledgments

Although this paper is self-contained it is the continuation of many previous ones in particular the one I just wrote before this one: "[Is the principle of contradiction a consequence of $x^2 = x$?]" (2016), related to a plenary talk I gave at the University of St Petersburg for the congress THE 12TH INTERNATIONAL CONFERENCE LOGIC TODAY: DEVELOPMENTS AND PERSPECTIVES in June 2016. Thanks again to my Russian colleagues for the invitation, in particular to Elena Lisanyuk and Ivan Mikirtumov.

I had the original idea of the present paper in July 2016 when in the Island of Santorini in Greece, formerly known as *Kallíste* (Καλλιστη, "the most beautiful one") and fictionally as ***Atlantis***, (Ἀτλαντὶς νῆσος, "the island of Atlas"). I was invited on the Island by Ioannis Vandoulakis, organizer of the event THE LOGICS OF IMAGE: VISUALIZATION, ICONICITY, IMAGINATION AND HUMAN CREATIVITY.

Following the idea of this congress I made extensive use in this paper of images, similarly as in recent papers, in particular "Possibility, Imagination and Conception" (2016) that I presented at this event. This is related to a project I am developing to promote the use of images in philosophy, *The World Journal of Pictorial Philosophy* : www.wjpp.org

I did not expound the present "MANY 1" paper at the event but had the occasion to discuss some of its contents with Ioannis who is a specialist of history of mathematics, and with other participants of the event, in particular Dénes Nagy the president of the *The International Society for the Interdisciplinary Study of Symmetry*, former student of the great Hungarian historian of mathematics, Árpád Szabó (whose work I know since my youth), and the plastic artist Catherine Chantilly.

I would like also to thank Mihir Chakraborty, founder of the *Kolkata Logic Circle*, who invited me to write this paper for a volume dedicated to pluralism in mathematics. I know

Mihir since a couple of years. He was an invited speaker at the 3rd CONGRESS ON THE SQUARE OF OPPOSITION we organized in Beirut in June 2012 and he gave a tutorial at the 4th WORLD CONGRESS AND SCHOOL ON UNIVERSAL LOGIC we organized in Rio de Janeiro in April 2013. After that I have organized with him the 5TH WORLD CONGRESS ON PARACONSISTENCY in February 2014, at the *Indian Statistical Institute in Kolkata*, whose motto is "Unity in diversity". Mihir also invited me to take part to another event in Kolkata just after this one: INTERNATIONAL CONGRESS ON HISTORY AND PHILOSOPHY OF MATHEMATICS - TRIBUTE TO SIR ASHUTOSH MOOKHERJEE where I presented the talk "Bourbaki and Modern Mathematics", which I never transformed into a paper but some things I said there are included in the present paper.

I was immersed in a Bourba-très-chic atmosphere since my youth but I deepened my knowledge about the history and philosophy of Bourbaki when in São Paulo, Brazil, in 1991-1992 working with Newton da Costa who used to take me to the house of his former teacher Edison Farah (1911-2006), the host of André Weil, Jean Dieudonné and Alexandre Grothendieck during their frequent visits to the University of São Paulo in the 1940s and 1950s. Farah proved a conjecture that Weil thought was false: general distributivity of conjunction relatively to disjunction is equivalent to the Axiom of Choice. Therefore one more formulation of AC among many ones.

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