Monosequent Proof Systems

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abstract. In this paper we present a sequent sequent with monosequents (sequents with only one formula on each side) for the logic of conjunction and disjunction. We first start by a general discussion about sequent systems. We then introduce the system $\mathbf{S11}_{\wedge,\vee}$ and prove some basic results, in particular the fact that distributivity does not hold. In a following part we show that the logic generated by this system is indeed the fibring of conjunction with disjunction. We furthermore discuss the relation between this system and lattice theory. We end up with the story of the development of this work and some personal recollections.

Dedicated to Amilcar Sernadas for his x-th birthday $(0 \leq x \leq \omega)$

Navegar é preciso, viver não é preciso.
Fernando Pessoa

1 Historico-philosophical remarks on systems of sequents

Systems of sequents are proof systems due to Gerhard Gentzen (1935). The expression “sequent calculus” (plural “sequent calculi”) is nowadays often used, but was originally not used by Gentzen himself. In his seminal work *Investigation into logical deduction*, published in two parts ([Gentzen, 1935a] and [Gentzen, 1935b] (translated in French in 1955 with useful comments [Gentzen, 1955] and in 1969 in English [Gentzen, 1969]), Gentzen introduced a particular use of the word “sequent” (in German: *Sequenz*), but he used neither the expression “systems of sequents”, nor “sequent calculus”.

In this 1935 work, where sequent systems are introduced, he presents two kinds of systems both with a classical version and an intuitionistic version. He uses the expression “Calculus of natural deduction” (*Kalkül des natürlichen Schließens*) for the first kind of systems, hence the acronyms NJ and NK (J stands for *intuitionistischer* and K stands for *klassischer*, the transformation of “j” into “i” has been attributed to a misreading of the typographer); and he uses the expression “Logistic calculus” (*logistischer Kalkül*) for the second type, hence the acronyms LJ and LK.

The expression “systems of propositions” (*Satzsysteme*) was used by Paul Hertz for systems from which Gentzen’s sequent systems are directly inspired. These systems were presented in a series of papers all written in German: [Hertz, 1922], [Hertz, 1923], [Hertz, 1928], [Hertz, 1929]. The first one has been written September 15, 1921 and published in *Mathematische Annalen* in September 1922, Volume 87, Issue 3, pp.246-269. It is entitled *Über Axiomensysteme für beliebige Satzsysteme. I. Teil.* It has been translated for the first time in English in the book *Universal Logic: an Anthology*, published in 2012 with the title *Axiomatic Systems for Arbitrary Systems of Sentences. Part I* with an introductory paper by Javier Legris entitled “Paul Hertz and the origins of structural reasoning” (Legris wrote several other papers about Hertz, see references in the bibliography of this latter paper, [Legris, 2012]).

The subtitle of this universal logic anthology is *From Paul Hertz to Dov Gabbay*. The reason to choose Hertz as the starting point of this anthology is the revolutionary position he undertook having a prophetic vision of logic where objects of reasoning are not restricted to sentences or propositions. Hertz uses the word “Satz” at the meta-level [1].

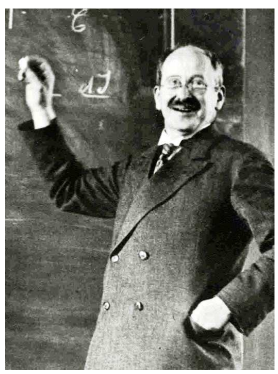
Hertz’s position is well summarized in Gentzen’s first paper entitled *Über die Existenz unabhängiger Axiomensysteme zu unendlichen Satzsystemen* [Gentzen, 1933] (“On the existence of independent axiom systems for infinite sentence systems”):

> A *proposition* has the form
> $$u_1, ..., u_n \rightarrow v.$$
> The u’s and v’s are called *elements*. We might think of them as events and the ‘proposition’ then reads: The happening of the events $u_1, ..., u_n$ causes the happening of v.

[1] Note that the word “Satz” in German is neutral relatively to the divide “sentence” / “proposition”, it can also mean *principle*, as in the expression *Der Satz vom zureichenden Grund* which is equivalent to *The principle of sufficient reason*.

> The 'proposition' may also be understood this: A domain of elements containing the elements $u_1, ..., u_n$ also contains the element v.
>
> The elements may furthermore be thought of as properties and then 'proposition' can then be interpretd thus: An object with the properties $u_1, ..., u_n$ also has the property v.
>
> Or we imagine the elements to stand for 'propositions', in the sense of the propositional calculus, and the 'proposition' then reads: If the propositions $u_1, ..., u_n$ are true, then the proposition v is also true..
>
> Our considerations do not depend on any particular kind of informal interpretations of the 'propositions', since we are concerned only with their formal structure.

This paper is about Hertz's systems, Gentzen does not introduce yet his own systems here but already coins a proper terminology, in particular the word "cut" that has become so famous.



Paul Hertz (1881-1940)

Gerhard Gentzen (1909-1945)

There are two reasons to qualify Hertz's work as *structural*. The rules introduced by Hertz are not about logical operators such as connectives or quantifiers. They have been called by Gentzen in 1935 "structural rules" (*Strukturschlußfiguren*). The second reason, connected with the first, is that Hertz anticipated the idea of a logic as a mathematical structure of the type $\mathcal{L} = \langle \mathbb{F}; \vdash \rangle$, where:

• $\mathbb{F}$ is a set of objects, called *formulas* (sets of formulas are called *theories*);

• $\vdash$ is a binary relation between theories and formulas, i.e. $\vdash \subseteq \mathcal{P}(\mathbb{F})\mathcal{X}\mathbb{F}$, called *consequence relation.* Let us call such kind of structure, an *abstract logic.* It is a structure of the same kind (but of different type) as an abstract algebra $\mathcal{A} = \langle \mathbb{A}; f_{(i \in I)} \rangle$ where:

• $\mathbb{A}$ is a set of objects;

• $f_{(i \in I)}$ is a family of functions.

Garrett Birkhoff promoted *Universal Algebra* as the study of the class of all abstract algebras ([JYB, 2015s] for more details and bibliographical references). By analogy we have promoted *Universal Logic* as the study of abstract structures of the type $\mathcal{L} = \langle \mathbb{F}; \vdash \rangle$ (see [JYB, 1994] and [JYB, 1995]).[2]

[2]The reason to use "$\mathbb{F}$" and not "$\mathbb{L}$" as the name of the domain is to avoid to make a connection with *language*. The elements of the domain of a logic are objects that can be events, etc. (in the spirit of Hertz's approach). Formulas are at best names for these objects.

According to this approach logical operators have to be understood from the point of view of a more fundamental notion (cf. [Koslow, 1992] and [Koslow, 2007]). Whatever name is given to her, what is important is the distinction between two strata. In Poland, Alfred Tarski (see [Tarski, 1928] and [Zygmunt, 2012]), not so much later, independently promoted a similar distinction, introducing the notion of consequence operator. What is similar in the work of Hertz and Tarski is that they have both developed a work at this pure level of abstraction. Tarski (jointly with Łukasiewicz, [Łukasiewicz-Tarski, 1930]) went on applying this abstract approach to logical operators, Hertz didn't, but Gentzen did it for him.
The notion of deduction (or inference) appears in a sequent system at three levels:

(1) as the connective "$\supset$" of implication,

(2) as the symbol "$\rightarrow$" at the middle of the sequent linking the antecedent formulas to the consequent formulas,

(3) as a horizontal line "——————" linking the premisses to the conclusion.

In a Hertz's system, in a Hilbert's system and in a Gentzen's natural deduction system it appears only at two levels. (1) does not appear in a Hertz's system. In the two other cases (1) generally appears but not (2).

Tarski's original theory of consequence operator has only one level (which corresponds to none of the three levels above). When applied to specific logical operators it has one more level, the level (1). It is important to emphasize that Tarski's theory is not a deductive system. Transitivity of the consequence relation/operator in Tarski's theory should not be confused with the cut rule of Gentzen. Such a confusion prevents to understand the cut-elimination theorem (for more on this topic see [JYB, 1999]).

2 The architecture of sequent systems

Let us first explain here why we prefer to use the expression "sequent system" than "sequent calculus". There are three reasons. Firstly, as rightly pointed out by Paul Halmos ([Halmos, 1970]), the word "calculus" is highly ambiguous, having different heterogeneous meanings. The semantical field of "system" is wide but less heterogeneous. Secondly, one of the dominating meanings of "calculus" is connected with "computing". The relation between proof theory and computability makes sense, but considering that mathematical proofs are totally reducible to some computable processes is highly controversial. The work of Gentzen has certainly contributed to show than in many ways mathematical proofs are computable, in particular one of the applications of the cut-elimination theorem is, in some cases, decidability. But on the one hand sequent systems are not necessarily leading to a reduction to computability and on the other hand they have virtues not limited to computability. The third reason, related to the second one, is to keep a connection with the perspective of Paul Hertz, who used the word "System".

For Hertz, a *Satz* is of the form:

$$a_1, ..., a_n \rightarrow b$$

Originally he was using a comma, which later on was withdrawn. Hertz is considering in fact that the antecedent is a set of objects, that he calls "complex" (*Komplex*) and that he represents by a upper case letter, e.g. 'K', the order having no importance despite the comma and natural numbers as subscripts. A rule for Hertz is therefore something of the form:

$$\frac{K \longrightarrow a}{L \longrightarrow b}$$

Gentzen transformed Hertz's *Satz* in somtehing he called a "sequent" (*Sequenz*). He kept the arrow at the middle, writing a sequent as follows ([Gentzen, 1935a], par 1, section 2.3):

$$\mathfrak{A}_1, ..., \mathfrak{A}_u \rightarrow \mathfrak{B}_1, ..., \mathfrak{B}_v$$

Gentzen was using Fraktur Gothic letters instead of lower case letters of the Latin alphabet. Fraktur Gothic letters are used to denote what he was calling "formulas", which are not anymore as for Hertz any kind of objects, but formulas of what is nowadays called "first-order logic". Lower case Latin letters were used, as it is nowadays used, as names for constants and variables of objects (by contrast to predicates / relations).

Something like $\mathfrak{A}_1, ..., \mathfrak{A}_u$ is a *sequence* of formulas in the usual sense of the word "sequence" in mathematics (there is an informal notion of sequence and a formal one developed in set theory, but the writing is the same). It means that the order is important and that $\mathfrak{A}, \mathfrak{A}$ is a sequence different from $\mathfrak{A}, \mathfrak{A}, \mathfrak{A}$. However Gentzen uses the word "sequence" (*Sequenz*) specifically for two sequences of formulas with at the middle Hertz's arrow of which he says: "the $\rightarrow$, like commas is an auxiliary symbol and not a logical symbol" ([Gentzen, 1935a], part 1, section 2.3). In English this idiosyncratic use of *Sequenz* has been translated by the word "sequent", a good choice to avoid ambiguity. Nevertheless it is good to remember that for Gentzen a sequent is a pair of sequences of formulas, that we could write:

$$< \Sigma_1 ; \Sigma_2 >$$

The semicolon ";" is a symbol which is used in mathematics for representing pairs. Dummett in his textbook on intuitionistic logic [Dummett, 1977] chose the nearby symbol ":" and does not used "$<$" and "$>$". This is not so nice but avoid the confusion with implication which is nowadays generally written using "$\rightarrow$" rather than "$\supset$" as it was done by Whitehead-Russell and Hilbert. It is anyway a better option than to use "$\vdash$" as it is frequently done generating a confusion between a sequent system and the logic it generates. Maybe it would be good to introduce an additional symbol for sequent like "$\succ$" (this what we have done in [JYB, 1999], see also [Humberstone, 2011], in particular section 1.21). But here we will use the original Hertz/Gentzen's writing, emphasizing the relation with Hertz's original work and also because we are using Sam Buss's LaTeX style file where this option has been chosen.

Gentzen considered the case of an empty sequence and he also famously considered a sequent system where the length of the right sequence is at most 1: the sytsem LJ for intuitionistic logic. And he has this surprising result according to which the difference between sequent systems for intuitionistic logic (LJ) and classical logic (LK) is structural in the sense that the rules for the logical operators are exactly the same, the difference being in the structure of the sequents. However in LJ the structural rules are also *mutatis mudandis* the same as in LK. At this point it is important to make a distinction between internal and structural determinations, structural principles being either on the internal side (structural rules) or on the external side (configuration of the sequents). This is described in the following table (from [JYB, 2001]).

	Internal Determinations	
External Determinations	Structural Rules	Logical Rules
	e.g.	
Sequent	Identity	(Dealing
-monosequent/multisequent		with the
-finite sets	Cut	morphology
-sequences		of the logic)
-structures, e.g.		
idempotent abelian semigroup		
	Weakening	
		e.g.
		Connectives
Rule		
-order on premises		Quantifiers
-cardinality of premises	Permutation	
		Modalities
	Contraction	
Proof		
-sequence/tree		
-length/depth	Associativity	
(finite or not)		
-order type		
Induced Logic		
e.g.		
$T \vdash a \Leftrightarrow \exists\ To$ finite, subset of T, $To \rightarrow a$ is derivable		
$T \vdash a \Leftrightarrow T \rightarrow a$ is derivable		
Structural Principles		

Table 1: The Architecture of Sequent Systems

It is also possible to reduce the size of the sequence to at most 1 on the left in a sequent, leaving the right sequence having a variable length. This has been proposed by Igor Urbas [Urbas, 1996] in a system he called LDJ, the "D" standing for "Dual". The logic generated by LDJ is a paraconsistent logic dual of intuitionistic logic (Note that not every paraconsistent logic is a dual of intuitionistic logic and that there are different dualizations of intuitionistic logic).[3]

[3]An interesting variation of sequents to deal with other non-classical logics are *hyper-*

What we are proposing in the present paper is to consider sequents where we have one and only one formula both on the left and on the right. Hertz already considered such kind of things, he called them "first-degree propositions". Gentzen renamed them *linear* propositions [Gentzen, 1933], but when introducing sequent systems he did not consider this configuration. We will call them here *monosequents*. The expression *first-degree propositions* would be ambiguous in particular because we are not calling here sequents propositions and the expression *linear sequent* would be ambiguous due to the use of the expression *linear logic* promoted by Jean-Yves Girard [Girard, 1987]. A *monosequent system* is a system of sequents where all sequents are monosequents.

The consequence relation of an abstract logic $\mathcal{L} = \langle \mathbb{F}; \vdash \rangle$ generated by a monosequent systems $S11$ is defined as follows:[4]

$$\boxed{T \vdash v \text{ iff there is a formula } u \text{ of } T \text{ such that } u \to v \text{ is derivable in } S11}$$

Let us consider the following sequent rule:

$$\frac{\mathfrak{A}, \Gamma \to \Theta \qquad \mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \to \Theta}$$

This is a photograph of the original left rule for disjunction presented by Gentzen ([Gentzen, 1935a] p.192). With the modern technology of LaTeX it is possible to produce something fairly similar:

$$\frac{\mathfrak{A}, \Gamma \longrightarrow \Theta \qquad \mathfrak{B}, \Gamma \longrightarrow \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \longrightarrow \Theta} \vee \text{ left}$$

In the present paper we will however replace the Fraktur Gothic letters by lower case Latin alphabet letters. We are therefore coming back to Hertz's original writing. This is not a problem since we are dealing only with propositional logic. Using this writing, the above rule is written as follows:

$$\frac{a, \Gamma \longrightarrow \Theta \qquad b, \Gamma \longrightarrow \Theta}{a \vee b, \Gamma \longrightarrow \Theta} \vee \text{ left}$$

And considering only monosequents we have:

$$\frac{a \longrightarrow c \qquad b \longrightarrow c}{a \vee b \longrightarrow c} \vee \text{ left}$$

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squents introduced by G.Pottinger [Pottinger, 1983] and A.Avron[Avron, 1987].

[4] Compare monosequents and this definition with what is called FMLA-FMLA framework in [Humberstone, 2011].

3 The monosequent proof systems $\mathbf{S11}_{\wedge,\vee}$ for the logic of conjunction and disjunction

3.1 Definition of $\mathbf{S11}_{\wedge,\vee}$

We consider in this paper only the monosequent proof system $\mathbf{S11}_{\wedge,\vee}$ for the logic of conjunction and disjunction.[5] We have on the one hand the axiom of identity and the cut rule, on the other hand logical rules for conjunction and disjunction. Both the axiom and the rules are schemes. a, b, c are any formulas.

AXIOM:

$$a \longrightarrow a$$

CUT RULE:

$$\frac{a \longrightarrow c \qquad c \longrightarrow b}{a \longrightarrow b}\,\text{cut}$$

LOGICAL RULES:

$$\frac{a \longrightarrow c}{a \wedge b \longrightarrow c}\,\wedge_{l1} \qquad \frac{b \longrightarrow c}{a \wedge b \longrightarrow c}\,\wedge_{l2} \qquad \frac{c \longrightarrow a \qquad c \longrightarrow b}{c \longrightarrow a \wedge b}\,\wedge_{r}$$

$$\frac{c \longrightarrow b}{c \longrightarrow a \vee b}\,\vee_{r1} \qquad \frac{c \longrightarrow a}{c \longrightarrow a \vee b}\,\vee_{r2} \qquad \frac{a \longrightarrow c \qquad b \longrightarrow c}{a \vee b \longrightarrow c}\,\vee_{l}$$

3.2 Proofs in $\mathbf{S11}_{\wedge,\vee}$

Let us see some examples of proofs in $\mathbf{S11}_{\wedge,\vee}$:

$$\dfrac{\dfrac{b \longrightarrow b}{a \wedge b \longrightarrow b}\,\wedge_{l2} \qquad \dfrac{a \longrightarrow a}{a \wedge b \longrightarrow a}\,\wedge_{l1}}{a \wedge b \longrightarrow b \wedge a}\,\wedge_{r}$$

$$\dfrac{\dfrac{\dfrac{a \longrightarrow a}{a \wedge (b \wedge c) \longrightarrow a}\,\wedge_{l1} \qquad \dfrac{\dfrac{b \longrightarrow b}{b \wedge c \longrightarrow b}\,\wedge_{l1}}{a \wedge (b \wedge c) \longrightarrow b}\,\wedge_{l2}}{a \wedge (b \wedge c) \longrightarrow a \wedge b}\,\wedge_{r} \qquad \dfrac{\dfrac{c \longrightarrow c}{b \wedge c \longrightarrow c}\,\wedge_{l2}}{a \wedge (b \wedge c) \longrightarrow c}\,\wedge_{l2}}{a \wedge (b \wedge c) \longrightarrow (a \wedge b) \wedge c}\,\wedge_{r}$$

$$\dfrac{a \longrightarrow a \qquad \dfrac{a \longrightarrow a}{a \wedge b \longrightarrow a}\,\wedge_{l1}}{a \vee (a \wedge b) \longrightarrow a}\,\vee_{l}$$

[5] $\mathbf{S11}$ has to be pronounced as "S-one-one" not as "S-eleven".

$$\dfrac{\dfrac{\dfrac{a\longrightarrow a}{a\wedge b\longrightarrow a}\wedge_{l1}\qquad \dfrac{\dfrac{b\longrightarrow b}{a\wedge b\longrightarrow b}\wedge_{l2}}{a\wedge b\longrightarrow b\vee c}\vee_{r1}}{a\wedge b\longrightarrow a\wedge(b\vee c)}\wedge_r \qquad \dfrac{\dfrac{a\longrightarrow a}{a\wedge c\longrightarrow a}\wedge_{l1}\qquad \dfrac{\dfrac{c\longrightarrow c}{c\longrightarrow b\vee c}\vee_{r2}}{a\wedge c\longrightarrow b\vee c}\wedge_{l2}}{a\wedge c\longrightarrow a\wedge(b\vee c)}\wedge_r}{(a\wedge b)\vee(a\wedge c)\longrightarrow a\wedge(b\vee c)}\vee_l$$

As an exercise the reader can perform proofs in $\mathbf{S11}_{\wedge,\vee}$ showing that:

$a\wedge a \dashv\vdash a \qquad a\vee a\dashv\vdash a$
$a\wedge b\dashv\vdash b\wedge a \qquad a\vee b\dashv\vdash b\vee a$
$a\wedge(b\wedge c)\dashv\vdash(a\wedge b)\wedge c \qquad a\vee(b\vee c)\dashv\vdash(a\vee b)\vee c$
$a\wedge(a\vee b)\dashv\vdash a \qquad a\vee(a\wedge b)\dashv\vdash a$

3.3 Metaproofs about $\mathbf{S11}_{\wedge,\vee}$

ATOMIZATION OF IDENTITY The axiom of identity can be atomized, i.e. it is possible to replace it by the following one where p is an any atomic formula:

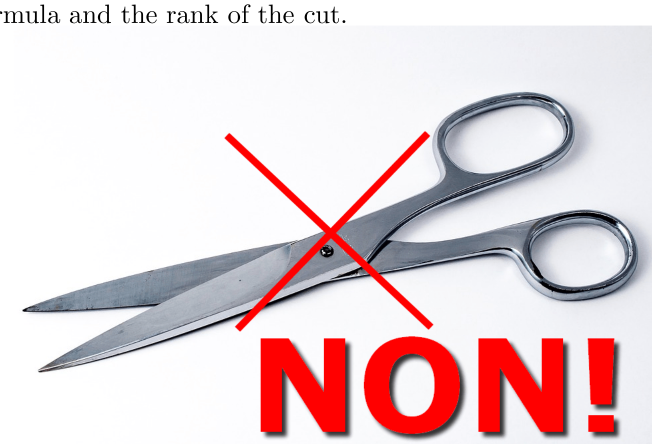
$$p\longrightarrow p$$

This is proved by recurrence, showing that it is possible to decrease the degree of the formula subject to identity. There are two cases: conjunction and disjunction.

$$\dfrac{\dfrac{a\longrightarrow a}{a\wedge b\longrightarrow a}\wedge_{l1}\qquad \dfrac{b\longrightarrow b}{a\wedge b\longrightarrow b}\wedge_{l2}}{a\wedge b\longrightarrow a\wedge b}\wedge_r$$

$$\dfrac{\dfrac{a\longrightarrow a}{a\longrightarrow a\vee b}\vee_{r1}\qquad \dfrac{b\longrightarrow b}{b\longrightarrow a\vee b}\vee_{r2}}{a\vee b\longrightarrow a\vee b}\vee_l$$

CUT-ELIMINATION Cut-elimination has been proved by Gentzen by double recurrence (maybe the first use of it): on the complexity of the formula and the rank of the cut. Let us see how we can in $\mathbf{S11}_{\wedge,\vee}$ lower the complexity of the cut formula and the rank of the cut.



Let us first examine the question of the *complexity* of the cut formula. We have only two cases. The cut formula is of the form $a \wedge b$ or of the form $a \vee b$. We examine the first case, and leave the other one, which is dual, for the reader. We have the following derivation:

$$\dfrac{\dfrac{u \longrightarrow a \qquad u \longrightarrow b}{u \longrightarrow a \wedge b}\wedge_r \qquad \dfrac{a \longrightarrow v}{a \wedge b \longrightarrow v}\wedge_{l1}}{u \longrightarrow v}\text{cut}$$

that we transform in the following derivation:

$$\dfrac{u \longrightarrow a \qquad a \longrightarrow v}{u \longrightarrow v}\text{cut}$$

The situation where the rule $\wedge_{l2}$ is used instead of $\wedge_{l1}$ is similar.

Now let us examine the question of the *rank* of the cut. We have six cases corresponding to the six logical rules we have. We examine only the case when the last rule used before the cut is $\vee_l$ and leave the other cases for the reader.

We have the following derivation:

$$\dfrac{\dfrac{a \longrightarrow c \qquad b \longrightarrow c}{a \vee b \longrightarrow c}\vee_l \qquad c \longrightarrow u}{a \vee b \longrightarrow u}\text{cut}$$

that we transform in following derivation:

$$\dfrac{\dfrac{a \longrightarrow c \qquad c \longrightarrow u}{a \longrightarrow u}\text{cut} \qquad \dfrac{b \longrightarrow c \qquad c \longrightarrow u}{b \longrightarrow u}\text{cut}}{a \vee b \longrightarrow u}\vee_l$$

DECIDABILITY As a corollary of cut-elimination we have the decidability of the logic generated by $\mathbf{S11}_{\wedge,\vee}$.

NON-DISTRIBUTIVITY
Using cut-elimination we can also perform a negative meta-proof, showing that there is no proof in $\mathbf{S11}_{\wedge,\vee}$ of the sequent (where a, b, c are atomic formulas):

$$a \wedge (b \vee c) \rightarrow (a \wedge b) \vee (a \wedge c)$$

The last used rule is either a $\wedge$ left or a $\vee$ right. Suppose that it is a $\wedge$ left left (i.e. $\wedge_{l1}$). Then we have the two following routes, which both lead nowhere:

$$\dfrac{\dfrac{a \longrightarrow a \wedge b}{a \longrightarrow (a \wedge b) \vee (a \wedge c)}\vee\text{ right}}{a \wedge (b \vee c) \longrightarrow (a \wedge b) \vee (a \wedge c)}\wedge\text{ left}$$

$$\dfrac{\dfrac{a \longrightarrow a \wedge c}{a \longrightarrow (a \wedge b) \vee (a \wedge c)}\vee\text{ right}}{a \wedge (b \vee c) \longrightarrow (a \wedge b) \vee (a \wedge c)}\wedge\text{ left}$$

Suppose now that it is a $\wedge$ right left (i.e. $\wedge_{l2}$). Then the next application of rule is either the $\vee$ left or a $\vee$ right. Consider that it is the the $\vee$ left. Then we have:

$$\dfrac{\dfrac{b{\longrightarrow}(a \wedge b) \vee (a \wedge c) \qquad c{\longrightarrow}(a \wedge b) \vee (a \wedge c)}{b \vee c{\longrightarrow}(a \wedge b) \vee (a \wedge c)}\ \vee \text{ left}}{a \wedge (b \vee c){\longrightarrow}(a \wedge b) \vee (a \wedge c)}\ \wedge \text{ left}$$

It easy to meta-prove that $b{\longrightarrow}(a \wedge b) \vee (a \wedge c)$ cannot be derived in $\mathbf{S11}_{\wedge,\vee}$. This is indeed also the case of $c{\longrightarrow}(a \wedge b) \vee (a \wedge c)$.

Consider now that the last application is a $\vee$ right, then we have the two following routes which both lead nowhere:

$$\dfrac{\dfrac{b \vee c{\longrightarrow}a \wedge b}{b \vee c{\longrightarrow}(a \wedge b) \vee (a \wedge c)}\ \vee \text{ right}}{a \wedge (b \vee c){\longrightarrow}(a \wedge b) \vee (a \wedge c)}\ \wedge \text{ left}$$

$$\dfrac{\dfrac{b \vee c{\longrightarrow}a \wedge c}{b \vee c{\longrightarrow}(a \wedge b) \vee (a \wedge c)}\ \vee \text{ right}}{a \wedge (b \vee c){\longrightarrow}(a \wedge b) \vee (a \wedge c)}\ \wedge \text{ left}$$

Coming back to the root of our meta-proof, we have now to examine the situation where the last application is a $\vee$ right. The two cases are symmetric. So we just consider the case of the $\vee$ left right. We have four roads leading nowhere:

$$\dfrac{\dfrac{\dfrac{a{\longrightarrow}a}{a \wedge (b \vee c){\longrightarrow}a}\ \wedge \text{ left} \qquad \dfrac{b \vee c{\longrightarrow}b}{a \wedge (b \vee c){\longrightarrow}b}\ \wedge \text{ left}}{a \wedge (b \vee c){\longrightarrow}a \wedge b}\ \wedge \text{ right}}{a \wedge (b \vee c){\longrightarrow}(a \wedge b) \vee (a \wedge c)}\ \vee \text{ right}$$

$$\dfrac{\dfrac{\dfrac{a{\longrightarrow}a}{a \wedge (b \vee c){\longrightarrow}a}\ \wedge \text{ left} \qquad \dfrac{a{\longrightarrow}b}{a \wedge (b \vee c){\longrightarrow}b}\ \wedge \text{ left}}{a \wedge (b \vee c){\longrightarrow}a \wedge b}\ \wedge \text{ right}}{a \wedge (b \vee c){\longrightarrow}(a \wedge b) \vee (a \wedge c)}\ \vee \text{ right}$$

$$\dfrac{\dfrac{a{\longrightarrow}a \wedge b}{a \wedge (b \vee c){\longrightarrow}a \wedge b}\ \wedge \text{ right}}{a \wedge (b \vee c){\longrightarrow}(a \wedge b) \vee (a \wedge c)}\ \vee \text{ right}$$

$$\dfrac{\dfrac{b \vee c{\longrightarrow}a \wedge b}{a \wedge (b \vee c){\longrightarrow}a \wedge b}\ \wedge \text{ right}}{a \wedge (b \vee c){\longrightarrow}(a \wedge b) \vee (a \wedge c)}\ \vee \text{ right}$$

At the end there is no way out, we cannot prove distributivity. It is also possible to show that we cannot prove the dual form of distributivity.

4 Fibring of conjunction with disjunction

In previous works we have been discussing combination of logics in particular the combination of the logic of conjunction with the logic of disjunction (see [JYB, 2004], [JYB, 2010], [JYB-Coniglio, 2011], [Humberstone, 2015]) . The basic idea of combination of logic crystallized as "fibring" by Dov Gabbay [Gabbay, 1999] is that nothing more should appear when putting two logics together. There must be no new things, product of an interaction. But it is not just a juxtaposition or a superposition, putting the two logics side by side or one on the top of the other. It is an intertwining, a weaving, ..., a mix, but a sterile mix. It is however not a pointless mix. Fibring is useful to construct or deconstruct step by step a logical structure [Carnielli et al., 2008].

The title of one of our papers is "A paradox in the combination of logics" [JYB, 2004]. In this note we point out that if we put together the logic of conjunction with the logic of disjunction combining valuations then we don't get fibring because we have distributivity, the same with standard sequent rules. Carlos Caleiro has however proved a general result in his PhD [Caleiro, 2000] according to which putting Hilbert-style rules together is not productive, i.e. that it leads to fibring. Here are some Hilbert rules for conjunction and disjunction (cf. [Marcelino-Caleiro, 2016]):

$$\frac{a\wedge b}{a} \qquad \frac{a\wedge b}{b} \qquad \frac{a \quad b}{a\wedge b} \qquad \frac{a}{a\vee b} \qquad \frac{a}{a\vee a} \qquad \frac{a\vee b}{b\vee a} \qquad \frac{a\vee(b\vee c)}{(a\vee b)\vee c}$$

We can use this set of Hilbert rules and Caleiro's general theorem to prove that $\mathbf{S11}_{\wedge,\vee}$ generates the fibring of conjunction and disjunction. It is enough to prove that the above set of Hilbert's rules $\mathbf{H}_{\wedge,\vee}$ is equivalent to $\mathbf{S11}_{\wedge,\vee}$.

To do that we can transform each rule of $\mathbf{H}_{\wedge,\vee}$ into a rule of $\mathbf{S11}_{\wedge,\vee}$. For example the first one is transformed into:

$$\frac{c \longrightarrow a \wedge b}{c \longrightarrow a} \wedge_{e1}$$

It is easy to show using the cut rule that $\wedge_{e1}$ and $\wedge_{l1}$ are equivalent.

$$\dfrac{c \longrightarrow a \wedge b \qquad \dfrac{a \longrightarrow a}{a \wedge b \longrightarrow a}\ \wedge_{l1}}{c \longrightarrow a}\ \text{cut}$$

$$\dfrac{\dfrac{a \wedge b \longrightarrow a \wedge b}{a \wedge b \longrightarrow a}\ \wedge_{e1} \qquad a \longrightarrow c}{a \wedge b \longrightarrow c}\ \text{cut}$$

The difficulty is how to derive the rule $\vee_l$ of $\mathbf{S11}_{\wedge,\vee}$ in (the sequent transposition of) $\mathbf{H}_{\wedge,\vee}$. A hint to prove this is to use lattice theory.

5 $\mathbf{S11}_{\wedge,\vee}$, lattice theory and logical algebra

The logic generated by $\mathbf{S11}_{\wedge,\vee}$ which is, as we have seen in the previous section, the fibring of the logic of conjunction with the logic of disjunction, has a very close connection to lattice theory. In some sense we can say: it is a lattice. And this is true, up to certain point. But approximate truth is easy, if not trivial (everything is true up to certain point ...), precise truth is another kettle of fish. Let's dive into the kettle.

First let us specify what a lattice is. If we want to compare two things, it is good to know what we have on both sides. There are a thousand ways to define a lattice. This variation of definition is a fascinating aspect of mathematics (Marshall Stone was surprised to see/prove that an idempotent ring is the same as a complemented distributive lattice, i.e. two different ways to define a so-called Boolean algebra). Are all these *different* ways the *same*? And are we able to precisely explain what "the same" means? There are different ways to do it, and they are not necessarily the same! For example we can say that two mathematical structures are "equivalent" iff they have a common expansion by definition up to isomorphism. This can be understood informally, in particular through some examples, but if we want to precisely formalize this, it is not so easy. We can do it in first-order model theory, but this theory is not so obvious, it took several decades until reaching a mature stage and it is only one way to do the job which has its own limitations.

Let us consider the following definition of lattice, which is the one given for example by Haskell Curry in his book *Foundations of mathematical logic* [Curry, 1963]:

A lattice is a structure with a partial order $\leq$ and two binary functions $\wedge$ and $\vee$ obeying the following axioms:[6]

$a \wedge b \leq a$ ($\wedge_{l1@}$)
$a \wedge b \leq b$ ($\wedge_{l2@}$)
if $c \leq a$ and $c \leq b$ then $c \leq a \wedge b$ ($\wedge_{r@}$)

$a \leq a \vee b$ ($\vee_{r1@}$)
$b \leq a \vee b$ ($\vee_{r2@}$)
if $a \leq c$ and $b \leq c$ then $a \vee b \leq c$ ($\vee_{l@}$)

If we write the axiom $\wedge_{r@}$ in a two dimensional figure, erasing the words, putting the two premisses up and the conclusion down, writing "$\rightarrow$" instead of "$\leq$", we have then exactly the rule $\wedge_{r\S}$ (we add here an "§" to the name of the sequent rule to better stress the contrast) of the sequent system $\mathbf{S11}_{\wedge,\vee}$; the same with $\vee_{l@}$.

If we want to operate similar transformations with the other axioms, we have first to replace them with equivalent axioms. Let us consider the axiom $\wedge_{l1@}$, we will leave the other cases as exercises. We replace this axiom by

if $a \leq c$ then $a \wedge b \leq$ c ($\wedge_{l1@\S}$)

Let us prove that $\wedge_{l1@}$ is equivalent to $\wedge_{l1@\S}$.

From right to left. Due to reflexivity of $\leq$, we have $a \leq a$, then by $\wedge_{l1@\S}$ we have the desired result: $a \wedge b \leq a$.

From left to right. If $a \leq c$, by $\wedge_{l1@}$ we have $a \wedge b \leq a$, then by transitivity we have the desired result: $a \wedge b \leq c$.

Let us compare these two proofs with the two following derivations in $\mathbf{S11}_{\wedge,\vee}$:

$$\frac{a \longrightarrow a}{a \wedge b \longrightarrow a}\,\wedge_{l1\S} \qquad \frac{a \wedge b \longrightarrow a \qquad a \longrightarrow c}{a \wedge b \longrightarrow c}\,\text{cut}$$

There are two differences between proofs in lattice theory and derivations within a sequent system. A first difference is that proofs in lattice theory, like proofs in mathematics in general, are performed in an informal way. It is not necessarily good to use the word "formal" to characterize such a difference, because it is quite ambiguous (see [JYB, 2008]).

We can write our two proofs in lattice theory in a similar way as the two above derivations in $\mathbf{S11}_{\wedge,\vee}$, writing "$\leq$" instead of "$\rightarrow$", "$\wedge_{l1@}$" instead of "$\wedge_{l1\S}$", "transitivity" instead of "cut". But doing that is not a sufficient condition to go from lattice theory to sequent system: although useful, it is not a necessary condition, because a sequent system does not reduce to such a way of writing, to a way of writing. Useful, but also dangerous, because we may have the illusion to have really brought about the transformation. A man can make up himself into a monkey, and someone may think he is a monkey. But this

[6]It would be absurd to write here "$\leq$" because the sign "$\leq$" is not itself a relation of order. If we want to use "$\leq$", we can say: "$\leq$" denote a relation of order. The same for "$\wedge$" and "$\vee$" which are names of binary functions.

is just an appearance. If we better examine the situation, we can see that the real transformation has not been achieved, that we are facing a fake monkey.

When doing proofs in lattice theory we are using a lot of resources. The idea of proof theory is to examine precisely how these resources work, capturing/describing them with a theory, namely proof theory. Proof systems such as sequent systems are part of proof theory but proof theory does not reduce to proof systems, it also includes the reasoning about these systems, which is itself generally carried on at an informal level, although it can also be formalized up to a certain point. In proof theory there is also a *meta* level. That is why it can be ambiguous to call proof theory "metamathematics", as it was done by Hilbert.

To avoid confusion we can use the word "derivation" for proofs within a sequent system, i.e. *formalized proofs*. This is the translation of the word "Herleitung" used by Gentzen, who considered it as a way shorther than "Beweisfigur" to speak about formalized proofs (cf 3.2. in [Gentzen, 1935a]). It is not clear that Gentzen is thinking here of short in the sense of the length of the word, because "Herleitung" has only one letter less than "Beweisfigur". This is rather a thought shortcut than a word shortcut.

When developing a proof system the use of the signs which are used is made quite precise. Gentzen's paper ([Gentzen, 1935a]) starts with a first section of five pages (178–182) entitled "Bezeichnungsfestsetzung." This 23 letter word has been wrongly translated in [Gentzen, 1969] as "TERMINOLOGY AND NOTATIONS". The French translation "Nomenclature des notations" in [Gentzen, 1955] is good.

This semiotic precision is often called "syntax". Considering the etymology of this word, "to put together in order', this is not absurd to use it. But syntax is often conceived by opposition to semantics. And some people want/ed to reduce semantics to syntax, considering that the meaning is fully given by "arrangement" of signs. What an "arrangement" is has to be clarified. Some people consider these arrangements as manipulations of signs, reducible to some writing devices. We can call this approach "syntactism". "Syntactism" has to be distinguished from a general position, that we can called "functionalism", according to which the meaning of a sign is given by the rule(s) to use it. A rule is not necessarily a writing device.

Let us now consider the second difference between proof in lattice theory and derivations within a sequent system. This difference is not, like the first one, directly about proof and derivation but about the objects dealt with. "$a \wedge b$" in lattice theory is not the same thing as "$a \wedge b$" in a standard sequent system. This latter difference is not a particularity of sequent system but of propositional languages in general. A propositional language is in general an absolutely free algebra, i.e. an algebra generated by some initial atoms, called "atomic formulas" (The word "formula" being used for any object of the structure, "molecular formulas" for formulas which are not atomic). In such an algebra the object denoted by "$a \wedge a$" is not the same as the one denoted by "a" even if $a \wedge a \dashv\vdash a$.[7]

In a lattice we have $a \wedge a \leq a$ and $a \leq a \wedge a$ and this means that $a \wedge a = a$,

[7]For more comments about that see [Anderson-Belnap, 1975], p.183 and [JYB, 1997].

i.e. that a and $a \wedge a$ are the same object; in other words: "a" and "$a \wedge a$" denote the same object. In logic we are not obliged to work with a language which is an absolutely free algebra, we can just work with an algebra, this is has been done by Roman Suszko (see [Jansana, 2012]). We can work with a sequent system with such a domain of formulas. In this case we are getting closer to lattice theory, but we are losing the possibility to make metaproofs on the complexity of formulas, because formulas don't have anymore an atomic ground. How to prove cut-elimination? Having a more complex structure, an absolutely free algebra, allows us to prove interesting results, having at the end in some sense the same structure. It is not exactly the same but were are able to precisely explain the relation between the two: we have on the one hand a logical structure, the fibring of conjunction with disjunction, on the other hand the factorization of this structure which is a lattice.

Two important remarks. Firstly what we have said is not completely exact. We have to consider a logic structure where there is only one object on the left of the consequence relation $\vdash$. The factorization is then obtained using the corresponding relation $\dashv\vdash$ which has also only one element on both sides. Secondly this procedure does not always work. A way to artificially make it work it to consider as logics, only structures where $\dashv\vdash$ is a congruence relation.

Let's examine more closely the situation where we have a sequent system on an arbitrary abstract algebra (not necessarily an absolutely free one). This situation has up to now not being investigated (however see [Negri-von Plato, 2004]). Suszko and his collaborators have studied a consequence operator on a whatsoever abstract algebra but this approach is different from the proof-theoretical approach. This approach is interesting for example to go on the opposite direction which generally prevails, algebraization of logic. It can be called logicization of algebra. Sequent systems are useful for that, in particular monosequent systems such as the one we have presented. But we can also work with monosequent systems where the intended meaning of the sign "$\rightarrow$" is identity, writing "=" instead of "$\rightarrow$". This logicization of algebra is different from *logicism*. It does not mean that we want to reduce algebra to logic, but the idea is to apply logical methodology to algebra. It is dual to *algebra of logic* conceived as application of algebraic methods to logic. Maybe the expression "algebra of logic" is better than "algebraization of logic" because it does not suggest a reductive approach. The expression "algebraic logic" is also not necessarily supporting the reductive stream. We can use the expression "logical algebra" for the kind of methodology we are proposing here. "Logic of algebra" would be too ambiguous.

If we have a sequent systems with sequent with "=" as a middle sign and objects of an arbitrary abstract algebra on both sides, we are closer to propositional logic than to first-order logic. This leads to a formalization of lattice theory proofs different to the one using the standard first-order sequent system LK. It is a formalization of the universal part of lattice theory, the one with only universal quantifiers. Putting then more structure on the algebra, it is possible to prove the decidability of this part of lattice theory, as an application of cut-elimination. This is not just a different transcription, it is a different conceptualization, leading to a different methodology with different results.

6 Development of this work and personal recollections

I don't remember exactly when I had the idea to work with monosequents, it is sometimes at the beginning of the 1990s. I wrote a short paper at this time with the proof that distributivity does not hold for this kind of systems, but I never developed and published this paper (I also wrote a short paper at the same time about logicization of algebra, that had the same fate). I decided to write the present paper to celebrate the anniversary of Amilcar, because it is connected with one of his favorite topics: combination of logics.

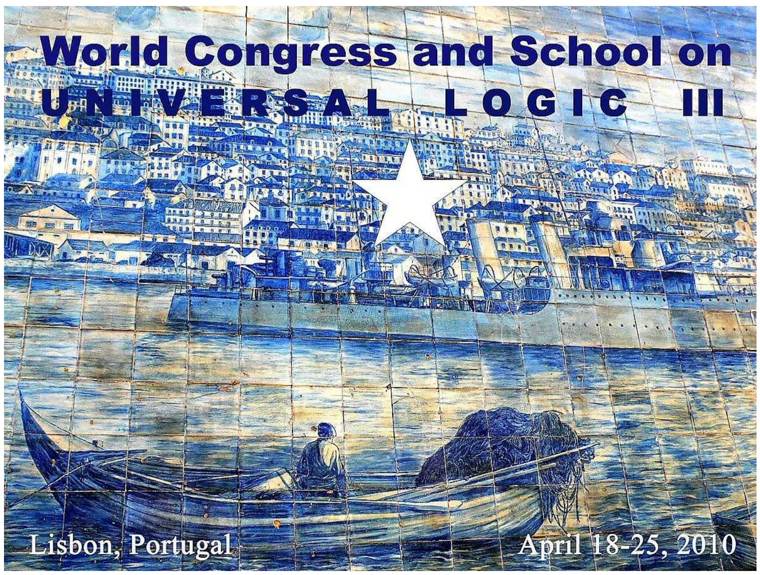
In 2004 Amilcar organized a congress on combination of logics in Lisbon (with the support of W.A.Carnielli). The event **ComBlog'04** – *Workshop on Combination of Logics: Theory and Applications* – took place where Amilcar is working, i.e. at the Department of Mathematics of IST (Instituto Superior Tecnico), Lisbon, Portugal. July 28-30, 2004. with many famous scholars including Dov Gabbay, Joseph Goguen, Jospeh Halpern, Dick de Jongh, Till Mossakowski, Don Pigozzi, Gabriel Sandu, Ventura Verdú, Frank Wolter. I presented there the talk "A paradox in combination of logic" [JYB, 2004] discussing the fact that when we put conjunction and disjunction together in a natural logical way appears an additional property. This additional property is in fact distributivity and I mentioned at this time, without entering in details, that a way to solve this paradox would be the use of monosequents.

I have been interested since many years to sequent systems. I studied this topic following a master class by Jean-Yves Girard in 1990 at the Department of Mathematics of the University Paris 7 and by myself studying the original work of Gentzen. Girard was quite fascinated by sequent systems and his work on linear logic [Girard, 1987] is much related with these systems. He also told us that the cut-elimination theorem was one of the most important results of modern logic (for him it was in particular in view of the Curry-Howard isomorphism, not from a more philosophical perspective, the transition from Aristotle's syllogistic to sequent systems operated by Hertz calling the cut rule "syllogismus" [Hertz, 1931] – about that see [JYB, 2017]).

I then wrote a Master's thesis on paraconsistent logic [JYB, 1990]. This work includes a sequent system for Newton da Costa's logic C1 ([da Costa, 1963]) as well a the cut-elimination theorem. Andrés Raggio, a former student of Paul Bernays, had tried to do that in the 60s. Later on, when in Wrocław, Poland, in 1993, I was in touch with Igor Urbas. He was doing research at the University of Konstanz, Germany, in a group directed by André Fuhrmann. I didn't met him, but one of my colleagues of Wrocław University, Tom Skura, was going there frequently. Urbas gave me a Konstanz's pre-print of his paper about *LDJ* (Dual-Intuitionistic Logic) [Urbas, 1996], where he studies a system of sequents with at most one formula on the left which generates a paraconsistent logic (Ubas did his PhD in Canberra on paraconsistency, see [Urbas, 1987]).

In 1994 I wrote my PhD in mathematical logic (defended in 1995, see [JYB, 1995]) proving in particular a theorem establishing a connection between sequent systems and valuations justifying an intuitive semantical reading of the rules of sequent systems on certain conditions This results justifies what Gentzen wrote on section 2.4. of [Gentzen, 1935a]: "The sequent $\mathfrak{A}_1, ..., \mathfrak{A}_u \rightarrow \mathfrak{B}_1, ..., \mathfrak{B}_v$ has exactly the same informal meaning as the formula

$\mathfrak{A}_1, ..., \mathfrak{A}_u \supset \mathfrak{B}_1, ..., \mathfrak{B}_v$." This *informal meaning* in my theorem is nothing else than the meaning given by the standard classical truth-tables for conjunction, disjunction and implication. This result has been inspired by the first paper by Gentzen which is on Hertz's systems [Gentzen, 1933] that I read at this time. My PhD is entitled *Recherches sur la logique universelle*, with subtitle *Excessivité, Négation, Séquents*. I coined the expression *Universal Logic* when in Poland in 1993 (see [JYB, 2015a]) as a name for a general theory of logics inspired by *Universal Algebra* as developed by Birkhoff (see [JYB, 2015s]).



I organized the 1st UNILOG (*World Congress and School on Universal Logic*) in Montreux, Switzerland in 2005 and the second in Xi'an, China in 2007. Amilcar liked very much the spirit of universal logic and strongly encouraged me to organize the third edition in Lisbon. We did that in 2010 with the support of his team, in particular Carlos Caleiro, also very much interested in the idea of universal logic. The 3rd World Congress and School on Universal Logic happened Abril 18-25, 2010 in Estoril, nearby Lisbon. In 2008 I started to prepare an anthology on universal logic [JYB, 2012], and Amilcar and Carlos agreed to write for this anthology a paper [Caleiro-Sernadas, 2012] presenting Dov Gabbay's "Fibred semantics and the weaving of logics" [Gabbay, 1996].

In 2008 I also moved back to Brazil from Switzerland and since then generally stop in Lisbon when travelling to Europe for a few days. So I had the opportunity to meet Amilcar on a regular basis. In 2012 Carlos launched together with João Marcos (from Natal, Brazil) a Marie Curie Exchange program between Brazil and Europe called **GeTFun** (*Generalizing Truth-Functionality*). I then spent prolonged stays in Lisbon developing contact with more Portuguese colleagues: Sérgio Marcelino, a former student of Carlos. and also Olga Pombo, director of the Center for Philosophy of Sciences of the University of Lisbon (CFCUL). Olga organized a worskhop entitled *The Place of Philosophy of Science at Lisbon University* on February 12-15, 2013, for commemorating the unification of the Techinal University of Lisbon with the University of Lisbon. I presented there the talk "Philosophy, Logic and Mathematics" [JYB, 2013]) and Amilcar presented a talk entitled O "triunfo do formalismo" ("The triumph of formalism") [Sernadas, 2013].

I had some discussions at this stage with Amilcar about this unification of the two main Universities of Lisbon and he was strongly supporting it (the institute he was working, the IST ,was part of the Technical University). I had also with him other discussions about various subjects, in particular during lunches at *El Corte Inglés* he kindly invited me to take part to with my wife Catherine and his wife Cristina. One of Amilcar qualities is that, beside being a very good logician, he has interest to think and discuss on all topics ... in a logical way! Being therefore a true universal logician.

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