Metalogic, Schopenhauer and Universal Logic

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Abstract. Schopenhauer used the word "metalogical" since his first work, *On the Fourfold Root of the Principle of Sufficient Reason* (1813), being the first to give it a precise meaning and a proper place within a philosophical system. One century later the word "Metalogic" started to be used and promoted in modern logic by the Russian logician Nicolai Vasiliev and the Polish School (Łukasiewicz, Tarski, Wajsberg). The aim of this paper is to examine the relations between the different uses of this word and doing that to try to have a better understanding of what Metalogic is and also logic *tout court*.

In a first section we examine and clarify the meaning of Metalogic in modern logic, comparing Metalogic to Metamathematics and Universal Logic. We make in particular a distinction between two trends in Metalogic that can be crystallized through *metatheorem vs. metaaxiom*.

In a second section we present Schopenhauer's use of the word, which is essentially through the notion of *metalogical truths*. We describe their locations within Schopenhauer's framework, standing side by side with other kinds of truths (metaphysical truths, logical truths, empirical truths), constituting altogether the Principle of Sufficient Reason (PSR) of Knowledge, one of the four roots of the PSR. We explain why Schopenhauer thinks that mathematical truths do not need to have a logical ground and present his view according to which metalogical truths are fundamental laws of thought that cannot be changed. We discuss the feminine nature he attributes to them and establish a parallel with Aristotle's vision of logic.

In a third section we examine how modern logic arose from a double challenge of the fundamental laws of logic: their reformulation and relocation, their relativization and rejection. We emphasize that this dynamic evolution was performed on the basis of some semiotical and conceptual changes at the heart of logic and Metalogic.

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Reason is of a feminine nature: it can give only after it has received. On its own, it possesses nothing but the empty forms of its own operation. Completely pure rational cognition gives us in fact only four things, the very metalogical truths.
Arthur Schopenhauer

Contents

Indiosyncralogical Schopenhauer

Arthur Schopenhauer (1788-1860) has been very popular during the second half of the 19th century and beginning of the 20th century, in particular among artists: Richard Wagner, Guy de Maupassant, Thomas Mann. Here is how he is nowadays presented by Mary Troxell in the *Internet Encyclopedia of Philosophy*: “Arthur Schopenhauer has been dubbed the artist’s philosopher on account of the inspiration his aesthetics has provided to artists of all stripes. He is also known as the philosopher of pessimism, as he articulated a worldview that challenges the value of existence. His elegant and muscular prose earn him a reputation as one the greatest German stylists.” [118]. This is a good summary of the general picture people have about Schopenhauer, in particular with no connection to logic.

Few people know that Schopenhauer had some interesting ideas about logic, at best they have heard about his essay on Eristical Dialectic generally known as *The Art of Persuasion*, but this has more to do with sophistry than logic itself. Schopenhauer was interested in particular in spheric representation of concepts, in the line of Leonhard Euler (1707-1783), and showed how we can go in this way from Good to Evil, from Paradise to Hell, and back, in a not so expensive way (See [73], a good starting point to explore Schopenhauer’s circus of conceptual circles).

Fig.1 Schopenhauer’s Spherical Thoughts

In the present paper we are dealing with a fundamental notion, Metalogic, examining what Schopenhauer said about that and comparing it with Metalogic as conceived in modern logic. Our objective is on the one hand to give a more open approach to the philosophical discussion about central concepts of modern logic, often reduced to contemporary problems without a general historical perspective (this is the case of the most famous books on philosophy of logic of the last decades, the one by Susan Haack published in 1978 [56]), and on the other hand to give a better vision of Schopenhauer, who had interesting views on many topics: the theory of colors (he was a friend of Goethe), biology (he knew the work of Lamarck), language (he knew many languages and translated Baltasar Gracián from Spanish to German), religion (he was the first Western philosopher to be interested in Oriental philosophy, both Hinduism and Buddhism), ... and also logic!

For this reason the present paper has been written in a way so that it can be of interest both for aficionados of Schopenhauer knowing few things about logic and logic lovers knowing quite nothing about Schopenhauer. We have given precise references both for the sake of rigor and as further readings for those wanting to know more.

Establishing a bridge between Schopenhauer and contemporary mathematical logic may look strange, not to say extravagant. But this is not so absurd if we consider that Schopenhauer developed some ideas about logic and also mathematics (connected to ideas of Ludwig Wittgenstein, 1889-1951, and Luitzen Egbertus Brouwer, 1881-1966) and that he is not so far away in time from modern logic (he was 66 year old when George Boole, at the age of 39, published the *Laws of Thought* [30]). And anyway it is good to try to have a general perspective, establishing connections between ideas of different times and origins, without being afraid to eventually fall into the sin of anachronism. We are not promoting sin (Chronos bless us), but, as they say in Germany, "no risk, no fun", and relating different things from different times is anyway tautologically anachronical.

We could have written a paper only about Metalogic according to Schopenhauer, but this would have been at best a good popular paper for lazy people having no time to read Schopenhauer. We cannot explain Schopenhauer better than himself. He was a philosopher who at the same time had his own vigorous original style and the capacity to write things rigorously, clearly and succinctly, showing that we can seriously write serious things without being boring (we will try to do the same here).

He wrote: "To use many words to communicate thoughts is everywhere the unmistakable sign of mediocrity. To gather much thought into few words stamps the man of genius." (PPA, V2, Ch23). And describing the general attitude of the philosopher he states that: "The real philosopher always looks for limpidity and precision, he will invariably try to resemble not a turbid, impetuous torrent, but instead a Swiss lake which by its calmness preserves transparency despite its great depth, a great depth revealing itself precisely through its great transparency." (4RP, §3)

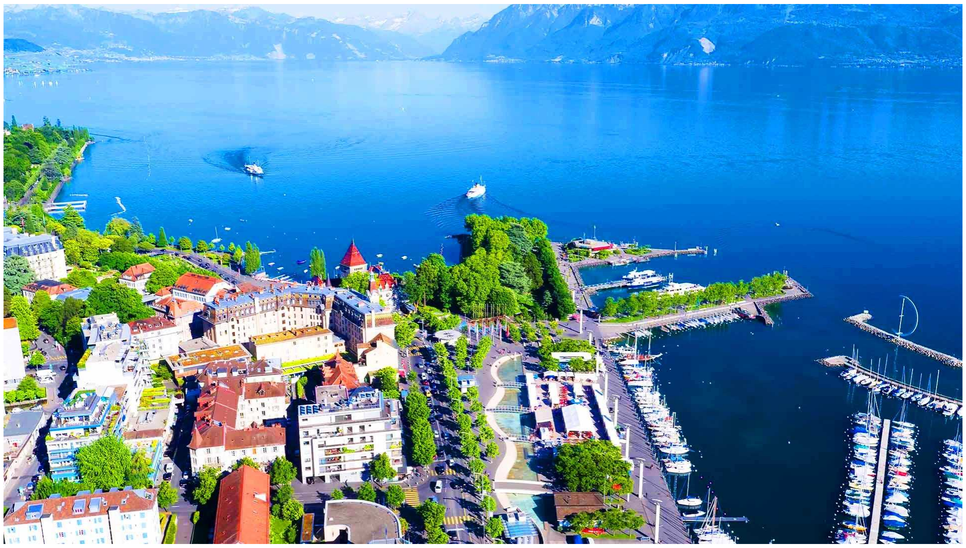


Fig.2 Swiss Lake and Mountains: Panoramic Transparency Ouchy and on the top left Montreux where was Organized the 1st World Congress on Universal Logic in 2005

This metaphor of the Swiss lake is interesting because Switzerland is a country not only with lakes but with high mountains and to be at the top of the mountain having a panoramic view is also the perspective of Schopenhauer who can be qualified as a "panoramic philosopher." This quality is described as follows by Friedrich Nietzsche (1844-1900) in his essay *Schopenhauer as Educator* (1874 [84]):

> His greatness is that he can stand opposite the picture of life, and interpret it to us as a whole: while all the clever people cannot escape the error of thinking one comes nearer to the interpretation by a laborious analysis of the colours and material of the picture; with the confession, probably, that the texture of the canvas is very complicated, and the chemical composition of the colours undiscoverable. Schopenhauer knew that one must guess the painter in order to understand the picture. But now the whole learned fraternity is engaged on understanding the colours and canvas, and not the picture: and only he who has kept the universal panorama of life and being firmly before his eyes, will use the individual sciences without harm to himself; for, without this general view as a norm, they are threads that lead nowhere and only confuse still more the maze of our existence. Here we see the greatness of Schopenhauer, that he follows up every idea, as Hamlet follows the Ghost, without allowing himself to turn aside for a learned digression, or be drawn away by the scholastic abstractions of a rabid dialectic.

Synopticity (in German: *Übersichtlichkeit*) was also promoted by Wittgenstein, much influenced by Schopenhauer, having read his books in his youth and once again after having written the *Tractatus* (see [82] and [80]).

In the perspective of this panoramic approach and to give an “avant-goût” of the content of our paper, we present the following chronological list of works we will talk about (not an exhaustive list, but a representative one):

- 1787, Immanuel Kant (1724-1804), 2nd edition of the *Critique of Pure Reason*
- 1813, Arthur Schopenhauer (1788-1860), *On the Fourfold Root of the Principle of Sufficient Reason*
- 1854, George Boole (1815-1864), *The Laws of Thought* [30]
- 1880, Charles Sandres Peirce (1839-1914), “A Boolian Algebra with One Constant” [87]
- 1881, John Venn (1834-1923), *Symbolic Logic* [122]
- 1913, Nicolai Vasiliev(1880-1940), “Logic and metalogic” [121]
- 1921, Emil Post (1897-1954), “Introduction to a General Theory of Elementary Propositions” [88]
- 1922, David Hilbert (1862-1943), “Neubegründung der Mathematik: Erste Mitteilung” [61]
- 1930, Jan Łukasiewicz (1878-1956) and Alfred Tarski (1901-1983), “Introduction into the sentential calculus” [77]
- 1937, Mordechaj Wajsberg (1902-194?), “Metalogische Beiträge. I” [125]
- 1952, Stephen Kleene (1909-1994), *Introduction to Metamathematics* [68]

But we will not follow in our paper a chronological order, neither forward, nor backward. Our itinerary is as follows: we start first with the conception of Metalogic in modern logic at the beginning of the 20th century, we then present Schopenhauer’s views on the Metalogical, and in a third part we treat the question of Metalogic through a panoramic analysis of the development of modern logic.

Our travel in time is to highlight the present to go ahead, not to relive the past or/and to spend happy vacations in the 19th century in Frankfurt am Main with Arthur Schopenhauer.

Nevertheless, according to the structure of our travel, Schopenhauer is at the center of our attention. It is the main course of our philosophical menu and at the very middle of this menu we have the section 2.2. entitled *Schopenhauer’s Theory of the Metalogical* which is the main dish. The whole menu should pamper both gourmets and gluttons.

1. Metamathematics, Metalogic and Universal Logic

The word "Metalogic" is a neologism combining the prefix "meta" with the substantive "logic". To understand the meaning of this combination in modern logic, we will start by examining another neologism, with the same prefix: "Metamathematics". This is a good point of departure because on the one hand the meaning of this neologism is quite clear and on the other hand the neologism "Metalogic" has been used in modern logic in particular under the influence of "Metamathematics".

1.1. Origin and Nature of Metamathematics

David Hilbert (1862-1943) made "Metamathematics" famous. He did not create the word but he was the first to give a precise meaning to it and to use it in a systematic way.[1] Before Hilbert the word was used in discussions about non-Euclidean geometry as a kind of synonymous to "Metageometry" which also was used, and Hilbert knew about that (for details see [128]).

But Hilbert started to use the word in a new way, as synonymous to another expression he promoted: "Proof Theory" (in German: *Beweistheorie*). The reason why is that for him the object study of Metamathematics are mathematical proofs, which are themselves the core of mathematics. A belief shared by many mathematicians. Nicolas Bourbaki (1935-1968) starts his famous multi-volume treatise, the Bible of modern mathematics [31], with the sentence "Depuis les Grecs qui dit mathématique dit démonstration" (literally: "Since the Greeks who says mathematic says demonstration"; inexact published translation: "Ever since the time of the Greeks, mathematics has involved proof"). The Greek prefix "meta" has different meanings but the way Hilbert is using it is *above*, in the intuitive sense that we study an object by being outside of it, upside being a good position, like when having a panoramic view at the top of a mountain.

Hilbert had the idea that Metamathematics was in some sense superior to mathematics, because it is the understanding of what mathematics is:

> The axioms and provable theorems, i.e. the formulae that arise in this interplay, are the images of the thoughts that make up the usual procedure of traditional mathematics; but they are not themselves the truths in an absolute sense. Rather, the *absolute truths* are the insights that my proof theory furnishes into the provability and consistency of these formal systems. [62]

The emphasis on "absolute truth" is ours. "Truth" in modern logic is often contrasted to "Proof", an opposition related to the contrast between *Model Theory* and *Proof Theory*.[2] But the use of "truth" Hilbert is doing here is not in the perspective of Model Theory (which did not exist at that time) but in the sense of a

[1] The same can be said about a central terminology and a central character of modern logic: "truth-value" and Gottlob Frege (1848-1925), see [20].

[2] This is also expressed as an opposition between semantics and syntax. "Proof Theory" as coined by Hilbert concentrates on mathematical proofs from a syntactical point of view, according to which mathematics, Hilbert says, "becomes a stock of formulae" [62]. "Model Theory" was coined by Tarski [114] and deals with the interpretation of the syntax, the *models* of the theories.

more fundamental and philosophical level, which is indeed the perspective of his Metamathematics.

Following the influence of Hilbert and his school, logic in the first half of the 20th century has been at some point identified with Metamathematics. Hilbert started to use word "Metamathematik" in the following two papers:

- "Neubegründung der Mathematik: Erste Mitteilung" (1922) [61],
- "Die logischen Grundlagen der Mathematik" (1923) [62].

As we can see, none of them has this word in the title. And there are no papers and books by Hilbert with this word in the title.[3] But Stephen Cole Kleene (1909-1994) published in 1952 a book entitled *Introduction to Metamathematics* [68], which was frequently re-edited (see Fig.3). This is an important textbook of modern logic which influenced a whole generation, as emphasized by Michael Besson in the forword of the re-printed 2009 edition: "Stephen Kleene was one of the greatest logicians of the twentieth century, and had an enormous influence on the subject. The book in your hands is the textbook that spread that influence far beyond his own students, to an entire generation of logicians." [70]

In this book there is no Model Theory. This is one of the main reasons why nowadays such a book is not considered as a serious introductory book to logic. Kleene himself explains/justifies the contents of this book in a paper entitled "The writing of *Introduction to Metamathematics*" [71].[4]

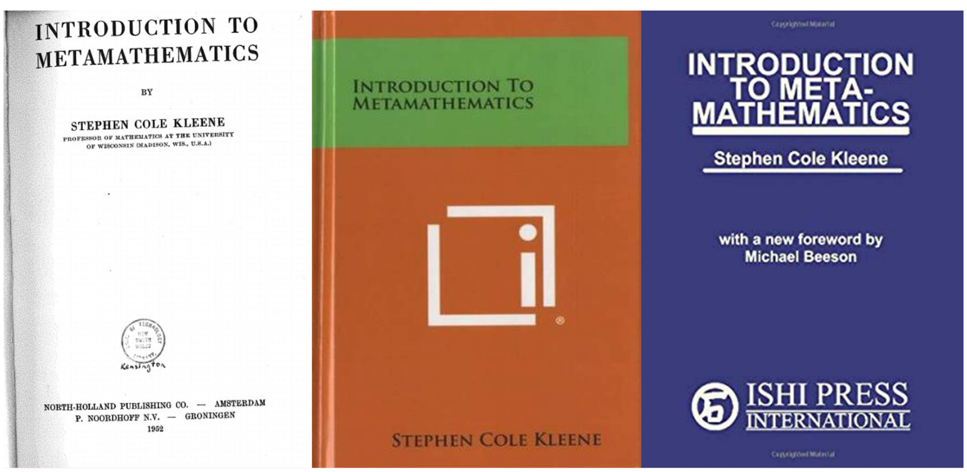


Fig.3 Different editions of Kleene's *Introduction to Metamathematics*

[3]The two books by Hilbert on logic are entitled *Grundzüge der theoretischen Logik* (1928), co-authored with Wilhelm Ackermann, 1896-1962 [63] (in English: *Principles of Mathematical Logic*) and *Grundlagen der Mathematik* (in English: *Foundations of Mathematics*), (Volume 1, 1934 - Volume 2, 1939), co-authored with Paul Bernays, 1888-1977 [64].

[4]Kleene later on (1967) published another textbook entitled *Mathematical Logic* [69] full of model theory, giving up Metamathematics both syntactically and semantically. In particular he uses in this book the expression "Model Theory" for propositional logic, which is up to now unfortunately not so common.

Kleene's famous book was developed under Hilbert's perspective of logic, often called "Hilbert's program", that we will not present here in details (to know more about it see [134]). But we will make some remarks explaining how "Metamathematics" can be understood in a way different than Hilbert's one and clarifying the relation between logic and Metamathematics in view of our discussion on Metalogic. We present a list of four points followed by comments.

(A) Metamathematics does not reduce to Proof Theory.
(B) The study of mathematical reasoning does not reduce to Proof Theory.
(C) Proof Theory does not reduce to Hilbert's formalist approach.
(D) Logic does not reduce to the study of mathematical reasoning.

(A) If we consider Metamathematics as a science having as object of study mathematics, there is no reason to reduce it to Proof Theory unless we reduce mathematics to proofs. There is also a whole conceptual and semiotical aspect. Boole in fact changed mathematics by considering operations on objects other than numbers or of geometrical nature. And Philosophy of Mathematics can be considered as part of Metamathematics.

(B) Moreover mathematical reasoning can be studied by other means than Proof Theory, in particular Model Theory, and also in psychological and cognitive perspectives.

(C) Hilbert is famous for having promoted Proof Theory using reduced means in particular regarding the question of finiteness. But it is possible to develop Proof Theory without such restrictions. Even in the school of Hilbert, Gerhard Gentzen (1909-1945) was not afraid to use infinistic methods, in particular for his famous proof of consistency of arithmetic using induction up to ϵ_0 [53].[5] The Polish school is famous not to have endorsed Hilbert's finitism and to have allowed the use of any mathematical tools, as for example the axiom of choice. This has been crystallized by the punny title of the book of Rasiowa and Sikorski: *The mathematics of metamathematics* (1963) [89].

(D) Logic can be defined as the study of all kinds of reasoning. We have to be careful since the word "logic" is also used to talk about reasoning itself (about this confusion see our paper "Logic is not logic" [17]). Although mathematical reasoning is important, this is not the only form of reasoning. Aristotle when developing the science of reasoning, in particular through a particular system, syllogistic, was considering reasoning about anything. This was also the case of the Stoics and later on of Boole himself, whose book title is *The Laws of Thought*, not *The Laws of Mathematical Reasoning*, nor *The Laws of Mathematical Thought.* But at some point in the development of modern logic people were focusing on mathematical reasoning: Peano, Frege, Whitehead, Russell, Hilbert...

[5] Gentzen also worked on "natural deduction", developing formal systems supposed to catch in a more natural way reasoning. For a discussion about that see [103], which makes a connection with Schopenhauer who was already concerned by this point.

1.2. From Metamathematics to Metalogic

As we have seen in the previous section, the word "Metamathematics" was promoted by Hilbert, identifying Metamathematics with Proof Theory. But the word "Metalogic" was not used by Hilbert, as explained by Haskell Curry (1900-1982), who studied in Göttingen and was one of the last students of Hilbert (see [40]): "Anyone who looks at all seriously at formalistic work of modern mathematical logic can hardly avoid noticing a great variety of words beginning with the prefix 'meta-'. One meets 'metalanguage', 'metasystem', 'metatheorem', 'metalogic', 'metacalculus', 'metasemiosis', and, in German, '*Metaaussagenkalkül*'. All these terms are described as in principle due to Hilbert. Actually the only one of them which Hilbert himself used is 'metamathematics'; the rest were invented by his followers on the basis of some analogy." ([43], pp.86-87)

At the beginning of the 20th century, "Metalogic" was used, independently and with different meanings by:

- the Russian logician Nicolai Alexandrovich Vasiliev (1880-1940)
- the Lvov-Warsaw Polish School (1915-1944)[6]

Vasiliev used this word before the people of the Lvov-Warsaw School and before Hilbert started to use the word "Metamathematics". In particular he published in 1913 a paper in Russian entitled "Logic and Metalogic". This paper was translated in English only 80 years later, in 1993 [121].[7] The view of Vasiliev on metalogic can be summarized by the following picture (Fig.4):[8]

EARTHLY LOGIC	ARISTOTELIAN, EMPIRICAL, THE LOGIC OF EARTH
IMAGINARY LOGIC	ANY LOGICAL SYSTEMS, APPLYING TO IMAGINARY WORLD OR OTHER PLANETS
METALOGIC	THE LOGIC OF THE FORM OF OUR THOUGHT, ABSTRACT AND NON-EMPIRICAL
THREE ASPECTS OF LOGIC - VASILIEV	

Fig.4 Vasiliev's Views on Metalogic

[6]To fix the ideas we have symbolically put here as dates of birth and death of this school respectively the coming of Łukasiewicz to Warsaw University and his departure from this university. Of course one can argue that this school started before 1915 and did not stop in 1944, that it is still alive, see the recent book *The Lvov-Warsaw School, Past and Present.* [50].

[7]See also two papers by Vasiliev of the same period: [119] and [120]. For a general presentation of Vasiliev and his work, see [2], [3].

[8]This figure is extracted from our previous paper "Is Modern Logic Non-Aristotelian?" [22] related to a lecture presented at a conference in honor of Vasiliev, October 24-25, 2012 at Lomonosov Moscow State University. And it was published in a book with other papers presented at this conference.

Here is the detailed explanation of Vasiliev of the nature of metalogic and its relation with other aspects of logic:

> I would call a logic without any empirical elements metalogic. The name "metalogic" is better suited for this discipline, as it indicates a formal analogy to metaphysics. Metaphysics is the knowledge of being regardless of the conditions of experience. Metalogic is the knowledge of thought regardless of the conditions of experience. Metaphysics is the science of pure being. It constitutes an abstraction from the world of phenomena, and it is the knowledge of that which is common to all empirical things. Metalogic is a discipline of pure thought. It is an abstraction from everything in thought that is empirical. There may be many worlds, but the essence of being is one. Such is the basic premiss of metaphysics. There may be Many logics, but they all have something in common which is only One, viz. metalogic. Metalogic, then, is the discipline of the formal aspect of thought regardless of its content. Therefore, the only formal logic is metalogic. [120]

We see that Vasiliev is using here the prefix "meta" by analogy with "Metaphysics". The word "Metaphysics" was originally used as the title of a book by Aristotle, based on the sense of the Greek prefix "meta" meanings "after" (different from the other sense we already talked about, "above"). This book was ordered in the corpus of Aristotle's work by commentators just after one entitled *Physics*, and since they were not able to find a proper name expressing the rather mysterious content of this book they decided just to name it *Metaphysics*. The meaning of the prefix "meta" Vasiliev is using is not an "afterward" syntactic meaning, it is related to the accidental semantics of the word "Metaphysics", which however essentially makes sense if we interpret "meta" as "above", Metaphysics being above experience.

According to Vasiliev's quotation it is quite clear that he did not use the word "Metalogic" under the influence of the pre-Hilbertian use of "Metamathematics" and the correlated use of "Metageometry". But there is a common background: Vasiliev's work was developed by analogy with the school of Non-Euclidean geometry (like Lobachevsky he was connected to Kazan): he created the expressions "Non-Aristotelean logic" and "Imaginary logic" (see [121]).

Let's see now how "Metalogic" was used in the Lvov-Warsaw School, about 15 years later, at the end of the 1920s, after Hilbert had started to use the word "Metamathematics". At the beginning of the paper by Łukasiewicz and Tarski "Introduction into the sentential calculus", originally published in 1930 in German with a Polish summary (see Fig.5), it is written:

> In the course of the years 1920-30 investigations were carried out in Warsaw belonging to that part of metamathematics – or better metalogic – which has as its field of study the simplest deductive discipline, namely the sentential calculus.

Why "Metalogic" is better than "Metamathematics"? To answer this question we have to examine what is the meaning of "Metalogic" in the Polish school. It is not that simple because there are two meanings which are entangled.

Odbitka ze Sprawozdań z posiedzeń Towarzystwa Naukowego Warszawskiego XXIII 1930. Wydział III.
Comptes Rendus des séances de la Société des Sciences et des lettres de Varsovie XXIII 1930. Classe III.

J. Łukasiewicz i A. Tarski.

Badania nad rachunkiem zdań.

Komunikat, przedstawiony przez J. Łukasiewicza dnia 27.III 1930 r.

Streszczenie.

W ciągu ostatnich kilku lat przeprowadzono w Warszawie badania z zakresu „metamatematyki", albo raczej „metalogiki" dotyczące najprostszej z pośród znanych obecnie nauk dedukcyjnych, mianowicie t. zw. rachunku zdań (teorji dedukcji). Celem niniejszego komunikatu jest zestawienie najważniejszych, przeważnie dotąd nieogłoszonych wyników, uzyskanych w toku tych badań.

J. Łukasiewicz und A. Tarski.

Untersuchungen über den Aussagenkalkül.

Vorläufige Mitteilung, vorgelegt von J. Łukasiewicz am 27.III 1930.

Im Verlaufe der letzten Jahre wurden in Warschau Untersuchungen durchgeführt, die sich auf denjenigen Teil der „Metamathematik", oder — besser gesagt — „Metalogik", beziehen, dessen Forschungsbereich die einfachste deduktive Disziplin, nämlich der s. g. Aussagenkalkül bildet. Die Initiative zu diesen Untersuchungen geht auf Łukasiewicz zurück; die ersten Ergebnisse rühren von ihm sowie von Tarski her. Im Seminar für mathematische Logik, das seit 1926 an der Universität Warszawa von Łukasiewicz geleitet wird, wurden auch die meisten der unten erwähnten Ergebnisse der Herren Lindenbaum, Sobociński und Wajsberg gefunden und besprochen. Die Systematisierung aller dieser Ergebnisse und die Präzisierung der einschlägigen Begriffe stammt von Tarski.

Fig.5 Łukasiewicz and Tarski on Metalogic

The first meaning is the most common one and is directly related with other meta-words, in particular "meta-theorem". There is a logic system, for example sentential logic, that must be differentiated from the study of this system, which is *Metasentential Logic*. This expression is explicitly used by Mordechaj Wajsberg (1902-194?) in a paper using this very word in the title: "Beiträge zum Metaaussagenkalkül" (1935) (in English: "Contributions to metasentential logic") [125]. This is part of Metalogic, as well as the study of other logical systems.

If we consider Classical Sentential Logic presented as a so-called "Hilbert system", there are some axioms and rules, from which some theorems are derived. For example $p \to p$ is such a theorem. This a theorem of the system. But there are also theorems about this system, for example the decidability of this system. Such a theorem is called a "metatheorem".

In general we don't make the difference between a system or a theory and the study of this system or theory. Because the two come together. A theory about the physical world, like the theory of relativity is called a "physical theory", not a "metaphysical theory", although this could make sense if we consider that it is about the physical world. And the study of the theory of relativity is also not called a metaphysical theory, in particular because it is not clearly distinguished from the theory itself.

Now let's consider the theory of natural numbers. It is standardly called "number theory" and its objects of study are the natural numbers. And there is something which is called *Peano arithmetic*, bearing the name of the famous Italian logician Giuseppe Peano (1858-1932). This theory reduces to a small group of axioms, like "every number as a successor", which is formally expressed in First-Order Logic (the main logical system of modern logic) as $\forall x \exists y s(x)=y$. The intended meaning of "s" is successor, but this meaning has to be specified by the formal apparatus. Due to the basic framework of First-Order Logic it is a unary function. We have then another axiom stating that this function is injective, expressing the fact that a number cannot have two successors, shaping the concept of successor into immediate successor, etc.

Peano arithmetic is a theory about the theory of numbers, describing reasoning about these numbers. This is mathematical reasoning. For this reason, Peano arithmetic can be considered as part of Metamathematics. Now what about the study of Peano Arithmetic? Is it Metametamathemtics? Generally it is simply considered as part of Metamathematics, because no clear distinction is made between Peano Arithmetic and the study of it. Let us consider a theorem such as Euclid's theorem, according to which there is infinitely prime numbers. The formalization of this theorem in Peano Arithmetic can be considered as part of Metamathematics, but this formalized theorem is not considered as a metatheorem. Here are some "real" metatheorems about arithmetic:

- The two Gödel's incompleteness theorems, 1932 [54]
- Existence of non-standard models of Peano Arithemtic, Skolem, 1934 [105]
- Relative consistency of arithmetics, Gentzen, 1936 [53]

We can say that these metatheorems are part of Metamathematics because they are results about a system describing mathematical reasoning. We cannot say the same about results about the incompleteness of a physical theory (cf. the work by Newton da Costa and Francisco Doria [39]) or the fact that classical physics can be axiomatized in first-order logic with only universal quantifiers (cf. the work by Rolando Chuaqui and Patrick Suppes [35]). A name, promoted by the very Polish School, that has been used for this kind of of research is "Methodology of Deductive Sciences".

If we consider the decidability of Classical Sentential Logic, we can say that it is part of the Methodology of Deductive Sciences, but can we say that it is part of Metamathematics? Is it a result about a system describing mathematical reasoning? Sentential Logic is a general system describing all kinds of reasoning, close to Stoic Logic (as shown by Łukasiewicz [76]). So what we have is a theorem about a logical system. That's why it makes more sense to consider it as part of Metalogic, rather than as part of Metamathematics. Many results of the Polish School are about Sentential Logic, classical or not, in particular the results of Wajsberg, who used *Metalogical Contributions* as the title of a work published in two papers (in 1937 [126] and in 1939 [127]). And if we have systems supposed to describe reasoning about physics or biology, like respectively those of Paulette Février (1915-2013) [49] and Joseph Henry Woodger (1894-1981) [133], both good friends of Tarksi, it is better to call the study of these systems "Metalogic" than "Metamathematics".

Emil Post, who proved the main and most important metatheorems about Classical Propositional Logic (functional completeness, completeness, decidability and maximality), simply used the expression "General Theory of Elementary Propositions" (cf. the title of his 1921 paper [88]).[9]

Tarski used "metamathematics" in several papers and, most important, as the title of his book gathering his pre-WW2 papers: *Logic, Semantics, Metamathematics - Papers from 1923 to 1938* [115]. But this was probably due to the strong influence of the Hilbert school which lasted during many years. This Tarski's book was published in 1956 only a few years after Kleene's book, which was dominating the market, as they say in the country of Walt Disney. However it is worth pointing out that there is only one occurrence of the word "Metalogic" in [115] (the one we have mentioned above) together with 4 occurrences of the adjective "metalogical".

Later on for the Polish edition of Tarski's work, Jan Zygmunt decided to use "Metalogika" as the subtitle of a book entitled *Logico-Philosophical Papers - Vol 2*, volume sharing many papers with [115]:

> We subtitle this volume *Metalogic* in an effort to sum up in one succinct word what we feel is most distinctive about the range of issues these works address. Tarski himself uses the terms 'metalogic' and 'metalogical' in various senses and contexts. He sometimes speaks of the metalogical conception of truth (and other notions). At other times he speaks of metalogic as a subject of

[9] "Sentential Logic" and "Propositional Logic" are both used. The first expression is used by people who want to emphasize, not to say to force, a syntactic or/and linguistic interpretation.

> study, or a field of research. He considers sentential calculi and their theories as properly belonging to metalogic. He also puts *Principia Mathematica* in this camp, together with set theory in all its multifarious versions. Tarski uses 'metamathematics' more often than 'metalogic' and 'metamathematical' more often than 'metalogical'.[135] [10]

But Zygmunt notes that Tarski used the word Metalogic on a review published in 1938 in the *Journal of Symbolic Logic*, saying that this work, which is Mostowksi's doctoral thesis:

> contains a succession of very valuable and interesting results from the domain of metalogic. As the subject of his research the author has chosen the system of *Principia Mathematica*, based on a simplified theory of types, and enlarged it by adding the axiom of infinity ... However, all the results obtained are, according to the author, applicable also to other kindred formal systems, in particular to the formalized system of Zermelo. [113]

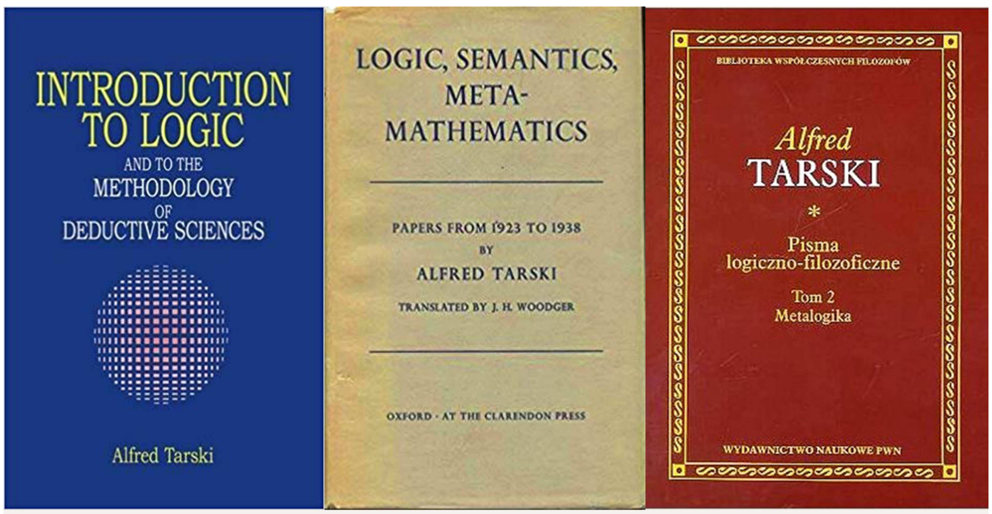


Fig.6 Three books of Tarski with different terminologies

If we prove theorems about propositional logic using mathematics, we can say that we are doing mathematics, mathematics of logic, or mathematical logic (this last expression is ambiguous because it can also be interpreted as "the logic of mathematics"). But what kind of mathematics is it exactly, is it a special mathematical theory? Most of the time Metalogic is developed in an informal way, like standard mathematics. But one may work on that, develop a theory about that, this is what people have done in Poland, in particular Alfred Tarski with the theory of consequence operator. Then we go to another sense of Metalogic that will be the main treat of our next section.

[10]Translation from Polish courtesy of Robert Purdy – checked and revised by Zygmunt.

1.3. From Metalogic to Universal Logic

In the previous section, we have seen two different and independent uses of the word "Metalogic", one due to Vasiliev, in between the Geometrical use of "Metamathematics" and Hilbert's use, one due to the Polish school, following, and inspired by, Hilbert's Metamathematics. In this section we will see another aspect of the notion of Metalogic emerging from the Polish school and that can be related to Vasiliev's notion. An aspect that also will be a useful bridge between the modern understanding of Metalogic and Schopenhauer's notion of metalogical truth.

After Hilbert's Metamathematics, we have many meta-words, as pointed out by Curry. But the word "meta-axiom" is not common at all. Difficult to find some occurrences of it, if any. On the other hand it would make sense to call Tarski's axioms for the consequence operator "meta-axioms", if we consider that these axioms are axiomatizing the so-called "axiomatic systems" (see [15]).

"Axiomatic systems" in logic is an expression used to qualify proof-theoretical systems in which there are lots of axioms and few rules. These systems are also often called "Hilbert systems" because Hilbert used to use them, but he was not the first and the only one. Such systems were also promoted by Whitehead and Russell in *Pincipia Mathematica* [130]. Let us emphasize that a Hilbert system with many axioms and only one rule can equivalently be presented with many rules and few axioms (see e.g. [90]). And this is not the essential feature of these systems by contrast with other proof-theoretical systems, such as Gentzen's sequent systems. The distinction between Hilbert and Gentzen's systems has to do indeed with Metalogic, since we can say that Gentzen's systems incorporate some metalogical principles at the logical level.[11]

Tarski's axioms for the consequence operator characterize some properties of the notion of logical consequence generated by such "Hilbert axiomatic systems", properties that are nowadays standardly called: "reflexivity", "monotonicity" and "transitivity". Curiously these properties are the same as the ones of a consequence relation model-theoretically (or semantically) defined by Tarski in his 1936's paper on logical consequence [110]. In the two cases we can call these properties "meta-properties", considering they are at the level of the infrastructure, part of a general theory of all existing or possible logical systems. They can be presented as follows:

- $T \vdash a$, when $a \in T$ (Reflexivity)
- If $T \vdash a$ and $T \subseteq U$, then $U \vdash a$ (Monotonicity)
- If $T \vdash a$ and $U, a \vdash b$, then $T, U \vdash b$ (Transitivity).

where "'a" and "b" denoted abstract objects intended to represent propositions, "T" and "U" denote sets of such objects, called *theories*, and "$\vdash$" a binary relation between them, called *consequence relation*.

These properties can be interpreted as proof-theoretical metaaxioms or model-theoretical metaaxioms for a consequence relation. Proof-theoretically, "$T \vdash a$" means that there is proof leading from T to a, model-theoretically, that if the

[11] For a presentation of the different kinds of proof-theoretical systems, the relation between them and their metalogical features, see[12].

propositions of which T is made are all true according to a given interpretation, then a is also accordingly true. We have used the symbol "$\vdash$", but Tarski was not using it.[12] This symbol has been used in a different way by Frege, Whitehead and Russell. Sometimes the symbol "$\vdash$" is used for proof-theoretical consequence by contrast to the symbol "$\models$" used by Tarski in the 1950s for the notion of model-theoretic consequence. The way we are using "$\vdash$" here is above this difference.

Such properties have been compared, and sometimes identified, with the *structural rules* of Gentzen's sequent systems directly inspired by what Paul Hertz, student of Hilbert, called *Satzsysteme* [59]. Hertz's rules look more like Tarski's axioms, belonging to the meta-level. And the common feature between Tarski and Hertz's approaches is that there is only one binary relation acting on some abstracts objects, no connectives or other logical operators (quantifiers, modalities, etc.) being specified. Tarski later on applied his theory of consequence operator to the study of specific logical systems by mixing the two in particular in his joint paper with Łukasiewicz ([77]). Hertz did not do that himself, this was however done by Gentzen who started his research activities by further developing Hertz's ideas. But first of all Gentzen developed a work about Hertz's framework staying at Hertz's general abstract level, proving a general completeness result [51], that can be viewed as establishing the link between the two Tarski's frameworks (but this was done independently of Tarski's work that Gentzen did not know). The next step for Gentzen was to apply Hertz's framework to the study of some particular systems, classical and intuitionistic systems. To do that he incorporated Hertz's rules as rules of his sequent systems, calling them "structural rules" [52].

There are two major ambiguities about the understanding of these structural rules. The first is that although structural rules are clearly distinguished from logical rules about logical operators. the two are at the same level (the same happens with Tarski's theory of consequence operator). The second is that structural rules may be confused with the properties of the consequence relations generated by these rules. Gentzen himself did not make a clear distinction between the two, because he did not explicitly consider the consequence relations generated by the rules of his sequent systems.

Serious misunderstandings may arise, a typical one is about the cut rule. The cut rule in a sequent system is a rule of the system. Gentzen with his famous *cut-elimination theorem* showed that the sequent system without this rule generates the same system as the system with the cut rule. These two systems generate therefore the same logics, having the same metalogical properties, in particular transitivity holds, transitivity being a property at the metalevel analogous to the cut rule. Gentzen's cut-elimination theorem can be properly understood only by

[12] Moreover Tarski was not considering a relation but a function, an "operator" acting on theories. It seems that for developing his theory he was influenced by the topological work of Kuratowski, with whom he collaborated at some point. The three properties presented here are not the same as, but are equivalent to, the ones of Tarski's consequence operator which look like those of a topological space, see [107].

making a distinction between the two things, using two different names “cut” and “transitivity” and two different corresponding symbols.

Gentzen’s cut rule is taken from Hertz who did not use this terminology, but called it the “Syllogismus” [60], because for him that was a formulation of the Barbara syllogism, the canonical example of syllogism. Cut is indeed the basic mechanism of syllogistic, every syllogism is a cut: the middle term disappears, is “cut”. Surprisingly Gentzen showed that it is possible to exclude this mechanism from reasoning. He did that by presenting a system in which there is no cut in any logical rules. The cut mechanism is expressed and isolated in only one rule, the cut rule. Gentzen then showed that the cut rule is redundant, that it is possible to get the same results without using it. This is the consequence of his cut-elimination theorem, which is in fact a true metatheorem, one of the most impressive metatheorems of modern logic, both by the inner quality of its proof (using double recursion at the time when recursion theory was just starting), its philosophical value (seriously challenging Aristotle’s syllogistic), its numerous consequences (e.g. decidability) and applications (e.g. relative consistency of arithmetics).

The first publication about the consequence operator is a 1928 abstract in French entitled “Remarques sur les notions fondamentales de la méthodologie des mathématiques” [107] (Fig.7). Tarski presents the theory of consequence operator as part of methodology of mathematics, metamathematics, or methodology of deductive sciences (see [108], [109]). This last expression, which is the more general, is used in the title of the last paper where he stresses the following:

> For the purpose of investigating each deductive discipline a special metadiscipline should be constructed. The present studies, however, are of a more general character: their aim is to make precise the meaning of a series of important metamathematical concepts which are common to the special metadisciplines, and to establish the fundamental properties of these concepts. [109]

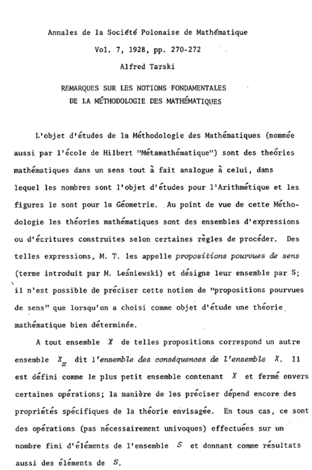
It is clear therefore that Tarski is conscious that he is opening another dimension. One could qualify this as “Metametalogic”. But this would be a bit monstrous.

In a project we have started to develop since the beginning of the 1990s [26], we have decided to use the expression “Universal Logic”. The choice of this terminology was in particular motivated by the analogy with “Universal Algebra”. Universal Algebra is a general theory of algebraic structures. The expression was coined by James Jospeh Sylvester (1814-1897), then used by Alfred North Whitehead (1861-1947), but its actual meaning is due to Garrett Birkhoff (1911-1996). Universal Algebra is a conceptual general framework for developing the study of any algebraic system. It could have been call meta-algebra, since the object of study of Universal Algebra are algebras.

But this proposal did not show up[13] and in some sense the meta way of speaking although it can be clear and meaningful, like in the case of “meta-theorem”, is

[13] Abraham Robinson (1918-1974) however talked about “The metamathematics of algebra” [91].

not so nice and sometimes confusing due to the most famous meta-word, "Metaphysics", which was accidentally chosen as we have already pointed out. The meaning of the word "Metaphysics" is not clear for two related reasons: the topics dealt with in Aristotle's book are difficult and abstract, the word itself does not express and/or explain what these topics are. Let us emphasize that in this book Aristotle is dealing with the principle of contradiction and we can say that Metalogic is part of Metaphysics in the sense of Aristotle if we consider Metalogic not just as a collection of technical results about logical systems but any examination and discussion surrounding, motivating, justifying, founding, these systems.



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Alfred Tarski

REMARQUES SUR LES NOTIONS FONDAMENTALES
DE LA MÉTHODOLOGIE DES MATHÉMATIQUES

L'objet d'études de la Méthodologie des Mathématiques (nommée aussi par l'école de Hilbert "Métamathématique") sont des théories mathématiques dans un sens tout à fait analogue à celui, dans lequel les nombres sont l'objet d'études pour l'Arithmétique et les figures le sont pour la Géometrie. Au point de vue de cette Méthodologie les théories mathématiques sont des ensembles d'expressions ou d'écritures construites selon certaines règles de procéder. Des telles expressions, M. T. les appelle *propositions pourvues de sens* (terme introduit par M. Leśniewski) et désigne leur ensemble par S; il n'est possible de préciser cette notion de "propositions pourvues de sens" que lorsqu'on a choisi comme objet d'étude une théorie mathématique bien déterminée.

A tout ensemble X de telles propositions correspond un autre ensemble X_x dit l'*ensemble des conséquences de l'ensemble* X. Il est défini comme le plus petit ensemble contenant X et fermé envers certaines opérations; la manière de les préciser dépend encore des propriétés spécifiques de la théorie envisagée. En tous cas, ce sont des opérations (pas nécessairement univoques) effectuées sur un nombre fini d'éléments de l'ensemble S et donnant comme résultats aussi des éléments de S.

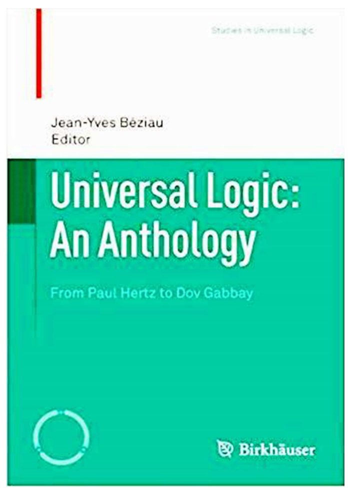


Fig.7 Tarski's paper included in the Anthology of Universal Logic

The fact that "Metaphysics" is quite confusing is the first reason not to use the word "Metalogic". The second reason is that "Metamathematics" is also quite confusing, due to the fact that it is much attached to Hilbert who used it in a very particular sense. There are therefore two reasons not to use the word "Metalogic", both related to the "meta" prefix. And then we have a third reason directly connected with this prefix: the idea is to reject the very use of a prefix.

In "Universal Logic", "universal' is not a prefix. It opens a new dimension, a new perspective, which is not only superposition. Semantically speaking the word "universal" is very powerful because it means at the same time unity and generality. However there is an ambiguity with "Universal Logic" since it can be interpreted and/or understood as a universal system of logic, which is an opposite view. This latter view corresponds to the spirit of *Principia Mathematica* or the work of Stanislaw Leśniewski (1886-1939) who was the PhD advisor of Tarski. But despite this inherent ambiguity we were not afraid to choose the expression "Universal Logic", in particular because the meaning of "Universal Algebra" as promoted by Birkhoff is quite clear, established and well-known, at least among mathematicians.

Moreover Universal Logic is directly inspired by Birkhoff's approach according to which there are no axioms. In both cases we are in the Realm of *Axiomatic Emptiness* (see [18]). We have the following parallel: in Universal Algebra, an *Abstract Algebra* is a set with a family of operators obeying no axioms, $A = \langle \mathbb{A}; f_i \rangle$, in Universal Logic, an *Abstract Logic* is a set with a consequence relation obeying no axioms, $L = \langle \mathbb{L}; \vdash \rangle$.

It was a way to make a clear demarcation from Tarski's approach to Metalogic which is based on (meta)axioms, and also from Vasiliev who, though he rejected the law contradiction from the sphere of his Metalogic, still considered that it consists of some basic fundamental principles.

In 2012 we published a book entitled *Universal Logic: an Anthology* [29] (Fig.7) including 15 items, chronologically classified, each one presented and commented by a specialist. Among them the papers by Hertz and Tarski we talked about and Part 6 is about Curry. It includes the two first chapters of his 1952 book *Leçons de logique algébrique* [42], translated in English and commented by his former student Jonathan Seldin. With this book Curry introduced for the first time the expression "Algebraic Logic". Ten years later he published a book which is a kind of extended version of the 1952 one with the title *Foundations of Mathematical Logic* [43] . This expression can also be seen as an alternative way to speak about Metalogic. Curry at the same time that he presents some technical tools, develops a lot of philosophical discussions which can properly be considered as part of Metalogic. The extracts of his French book presented in the Anthology of Universal Logic are in fact rather philosophical. Beside this Part 6 the only philosophical paper in this anthology is the one by Louis Rougier, "The Relativity of Logic" [95], that we will talk about in the third part of our paper after having presented Schopenhauer's ideas.

The aim of the Universal Logic project [26] is not only to develop Metalogic in the wide sense of a general theory of logical systems and structures but also to discuss and develop philosophical ideas related to such kind of theory and the basic concepts of logic. An expression such as "philosophy of logic" is not so good, as other expressions of the type "philosophy of ...", because it gives the impression of an afterward, as if philosophy would be comments on an already manufactured product. For this reason better not to use it if we think that philosophy is part of the production, a fundamental element of the conception, which can even be considered as the first stage.[14] There is also the expression "philosophical logic", but its meaning is even more confusing (see [106]) than the one of "mathematical logic" (that can be interpreted in two different non equivalent ways: *logic of mathematics* and *mathematics of logic*). We are therefore glad to welcome philosophical aspects of logic, including some of historical flavor, under the umbrella "Universal Logic" and that's why it makes sense to have the volume in which the present paper is included in the book series *Studies in Universal Logic.*

[14] Compare with what S.Haack says in the section *Logic, philosophy of logic, metalogic* of her 1978 book.

2. Schopenhauer's Theory of the Metalogical

As indicated by the title of his dissertation of 1813, *On the Fourfold Root of the Principle of Sufficient Reason* (in German: *Über die vierfache Wurzel des Satzes vom zureichenden Grunde*), Schopenhauer distinguishes four roots of the principle of sufficient reason (hereafter PSR). Here are they:

- PSR of becoming
- PSR of knowing
- PSR of being
- PSR of acting

We present them because it is important to have the general picture but we will not enter here in details for each of them. Our main interest is for the PRS of knowing where the metalogical is located. However it is important to say a few words about the PSR *tout court*, for people who have never heard about it, and also to present and discuss a bit the PRS of being, to have a proper understanding of Schopenhauer's vision of logic, considering its relation with mathematics. This is what we will do in the first section of this second part of our paper.

There are two versions of the essay of Schopenhauer, the original version of 1813 and a revised version in 1847. Further ideas about metalogical truth and logic can be found elsewhere in Schopenhauer's work. Here is the list including abbreviations we will use:

- *On the Fourfold Root of the Principle of Sufficient Reason* (4RP), 1813 and 1847
- *The World as Will and Representation* (WWR), 1818, 1844 and 1859.
- *Parerga and Paraliponema* (PPA), 1851.
- *Handwritten Manuscripts* (HWM), 1864, 1913.

And here is a list of the parts of these works especially relevant for our discussion:

- 4RP, Chapt V, §29. PSR of knowing
- 4RP, Chapt V, §30. Logical Truth
- 4RP, Chapt V, §32. Transcendental Truth
- 4RP, Chapt V, §33. Metalogical Truth
- 4RP, Chapt V, §34. Reason
- WWR, Vol.1, 1st Book, §9.
- WWR, Vol.1, 1st Book, §10.
- WWR, Vol.1, 1st Book, §15.
- WWR, Vol.1, Appendix: Criticism of the Kantian Philosophy
- WWR, Vol.2, Chpt IX. On Logic in General
- WWR, Vol.2, Chpt X. On the Science of Syllogisms
- WWR, Vol.2, Chpt XIII. On the Methods of Mathematics
- PPA, Vol.2, Chpt II. Logic and Dialectic
- HWM, Berlin lectures, 1820s, §Metalogical truth
- HWM, Eristical Dialectic, 1830s

2.1. The Tricky and Crutchy Euclid

The PRS is not an invention of Schopenhauer, his original contribution is to have distinguished four roots of the PRS. For many people the PRS is strongly connected or due to Gottfried Wilhelm Leibniz (1646-1716). But Schopenhauer in Chapter 2 of 4RP entitled *General survey of the most important views hitherto held concerning the principle of sufficient reason* of about 20 pages has only half a page about Leibniz (§9 of 4RP). Schopenhauer wrote the following: "Leibniz first put forth the principle of reason formally as a fundamental principle of all knowledge and science. He proclaimed it very pompously in many passages in his works, thereby even putting on airs about it, and portraying himself as if he were the first one to discover it; however, he knew nothing further to say about it, except that anything and everything must always have a sufficient reason why it is so and not otherwise, which must have been quite well known to the world before him."[15]

Schopenhauer quotes the French formulation of Leibniz of the PRS: "En vertu du principe de la raison suffisante, nous considérons qu'aucun fait ne saurait se trouver vrai ou existant, aucune énonciation véritable, sans qu'il y ait une raison suffisante, pourquoi il en soit ainsi et pas autrement". Leibniz also uses the Latin formulation *Nihil est sine ratione*, to which Martin Heidegger (1889-1976) gives much importance in his book *The Principle of Reason* (in German: *Der Satz vom Grund* [57]), a book in which Schopenhauer is strangely never mentioned. Schopenhauer does not focus on a specific linguistic formulation of the PRS. Although he considers that Leibniz was the first to put the PRS in the first place, Schopenhauer traces back the PRS up to Plato, quoting *Philebus* and *Timaeus* where Plato claims that everything which occurs, occurs with a cause, and then criticizes Aristotle and more generally the classical philosophers: "We see that the Ancients still did not attain a clear distinction between the requirement (der Forderung) for a knowledge ground for founding a judgment (eines Erkenntnissgrundes zur Begrndung eines Urtheils) and that of a cause for the occurrence of a real event (einer Ursache zum Eintritt eines realen Vorganges)" (last paragraph of §6 of 4RP).

Schopenhauer is indeed the first to make a clear distinction between what we can call the epistemological and ontological versions of the PRS but he does not stop at the level of this simplistic dichotomy. He goes further on with a fourfold distinction. The ontological version is duplicated in two: the PRS of becoming concerning material phenomena (the law of causality) and the PRS of acting concerning human action. And so is duplicated the epistemological version: the PRS of knowing concerning knowledge in general and the PRS of being concerning *a priori* knowledge. The very name "PRS of being" is quite ambiguous and one may rather see it as an ontological version of the PRS. But Schopenhauer is a follower

[15] Maybe Schopenhauer is too harsh with the philosopher known for claiming that we are living in the best of all possible worlds, by contrast to Schopenhauer's idea, according to which we maybe are in the worst of all possible worlds. For a more neutral assessment of Leibniz on the Principle of Reason see [85].

of Kant on the question of the pure *a priori* intuitions. The PRS of being is ruling these intuitions, which according to Kantian philosophy are not reality, but conditions of apprehension of reality.

As explained in a paper I wrote many years ago [8], Schopenhauer is very critical to the use of logic in mathematics, because as a follower of Kant he believes in the grounding of mathematics in the pure *a priori* intuitions of space and time, on which geometry and arithmetics are, according to the Kantian perspective, respectively based (cf. §37, §38 and §39 of 4RP). Schopenhauer goes a step further than Kant by strongly insisting that mathematical truth therefore doesn't need logic. In particular he scapegoat Euclid:

> The principle of non-contradiction compels us to admit that everything Euclid demonstrates is true: but we do not find out why it is so. We have almost the same uncomfortable sensation people feel after a conjuring trick, and in fact most of Euclid's proofs are strikingly similar to tricks. The truth almost always emerges through a back door, the accidental result of some peripheral fact. An apagogic proof often closes every door in turn, leaving open only one, through which we are forced simply because it is the only way to go ... by our lights the Euclidean method can only appear as a brilliant piece of perversity (eine sehr glänzende Verkehrtheit). (WWR, §15)

Both Bouwer (cf. [47], [72]) and Wittgenstein (cf. Chapter 14 of [80]) have been strongly influenced by Schopenhauer's views of mathematics. But let us emphasize that for Schopenhauer logic is not leading us in the wrong direction, at the end we arrive at the same location. The point is that its "method" is an intricate path. Schopenhauer describes this with the following nice river metaphor:

> Euclid's logical way of treating mathematics is a useless precaution, a crutch for sound legs ... it is like a night traveler who, mistaking a clear and solid path for water, takes care not to tread on it and instead walks along the bumpy ground beside it, happy all the while to keep to the edge of the supposed water. ((WWR, §15)

Schopenhauer says that Euclid uses "intuitive evidentness to support only what he absolutely had to (the axioms), supporting everything else with inference ... In mathematics, according to Euclid's treatment, the axioms are the only indemonstrable first principles, and all demonstrations are in gradation strictly subordinated to them." But according to Schopenhauer the theorems can also be supported by evidence and they don't need to be derived from the axioms: "every proposition again begins a new spatial construction. In itself, this is independent of the previous constructions, and can actually be known from itself, quite independently of them, in the pure intuition of space, in which even the most complicated construction is just as directly evident as the axiom is." ((WWR, §15)

The idea of Schopenhauer is that mathematical reasoning, whether in geometry or about numbers, does not need to be based on logic and that it is better to have mathematical proofs directly based on what is really supporting their truth, the pure *a priori* intuitions of space and time. He concludes Chapter 6 of 4PRS, devoted to the PRS of being, by saying: "I cannot refrain from again providing a figure which has already been given in other places, the mere appearance of which, without further discussion provides twenty times the conviction of the truth of the Pythagorean theorem than Euclid's mousetrap proof" and by providing the following picture (Fig.8):

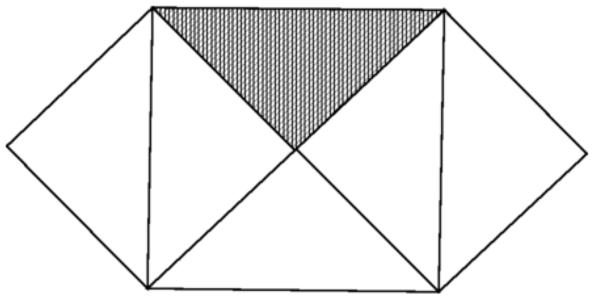


Fig.8 Pictorial Proof of the Pythagorean Theorem

A criticism that can be addressed to Schopenhauer is that visual reasoning on the one hand does not necessarily reduce to intuition of space, on the other hand does not only apply to space, it can be applied to anything. Reasoning involving colors for example can be developed (see [24], and for a general perspective see the multi-volume book *Proofs without Words* by Roger Nelsen [83]). This is not against Schopenhauer's examples of visual proofs, but it seriously challenges the space-to-space basis of his neo-Kantian philosophy of mathematics.

2.2. Metalogical Truths: Where they are and What they are

The PRS of knowing is about truth. Schopenhauer presents it as follows: "truth is the relation (Beziehung) of a judgment to something out of it, its sufficient reason." (4RP §39) There are four types of truth according to the kind of reason on which a judgment is based. The reason may be:

- a judgment (*formal or logical truth*)
- a sensible representation (*empirical truth*)
- a pure intuition (*transcendental or metaphysical truth*)
- the formal conditions of thought (*metalogical truth*).

Schopenhauer defines metalogical truth as follows: "a judgment may be founded on the formal conditions of all thinking, which are contained in the Reason; and in this case its truth is of a kind which seems to me best defined as metalogical truth." (4RP §33) In this essay Schopenhauer gives some formulations of these metalogical truths that we will present later on. Here we just list them with the names he gives to them in WWR (at the beginning of §10):

- identity
- non-contradiction
- the excluded middle
- PRS of knowing.

Here is the general picture (Fig. 9) describing the place of metalogical truths within the framework of the PRS:

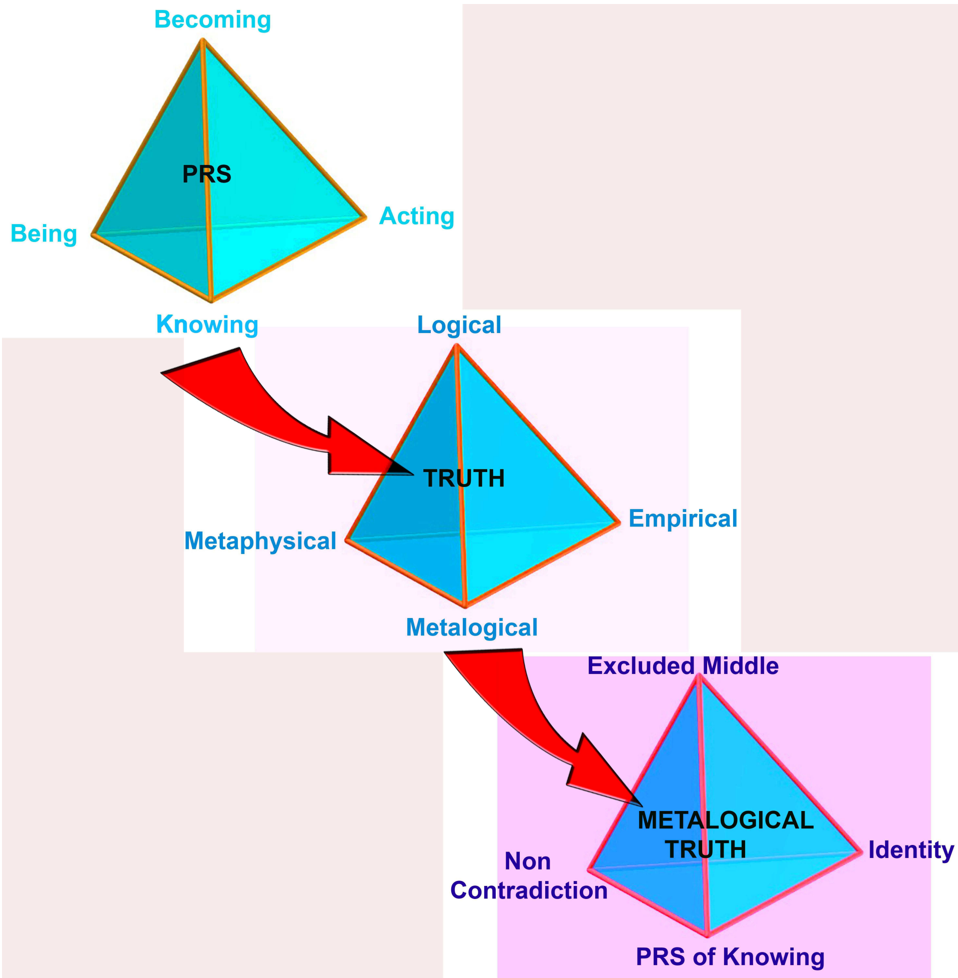


Fig.9 The Location of Metalogical Truth

As we can see Schopenhauer does not use "Metalogic" as a substantive, but as an adjective applied to truth. Metalogical is a quality of truth. We don't find the word "Metalogik" in his writings and he doesn't consider that there is a field of study corresponding to that.

Nevertheless we can talk about Schopenhauer's 'Theory of the Metalogical". By that we mean his views on metalogical truths, where they are and what they are and the general philosophy explaining / justifying that. We have worked up to now on their position within the general Schopenhauer's PRS framework. If we want to have a better understanding of what they are we have to go upstream and downstream.

- Upstream: what are the basis of the metalogical truths, how we know them, why they are four and why they are these fours?
- Downstream: what does arise from these four metalogical truths, in which sense are we using them, what are their relations with reasoning?

The reason why there are exactly four metalogical truths seems a bit artificial. As well shown by Fig.9, Schopenhauer has a general systematic 4-scheme. His main book *The World as Will and Representation* also has 4 parts (which do not correspond to the 4 roots of the PRS). We can say that Schopenhauer is often following a kind of 4-ideology, not to say 4-mysticism (see [97]), by contrast to Georg Wilhelm Friedrich Hegel (1770-1831), or Peirce later on, praising the 3. The three first metalogical truths are not due to Schopenhauer, we will come back to their formulations and meanings in the third part of our paper. To add the PRS of knowing as a fourth metalogical truth is a bit weird: the PRS of knowing is the fourth part of itself. This is somewhat circular, like a dog biting is tail. Anyway it allows to square everything ... Let us emphasize that Schopenhauer considers metalogical truths as judgments, and that therefore the PRS of knowing is a judgment about judgments, a "metajudgment". Schopenhauer is not using this word but this would be a reason to "metafy" it.

Later on, in the supplements of WWR, Schopenhauer proposed a reduction of these four metalogical truths to only two: on the one hand the PRS of knowing, on the other hand the condensation of the three other ones in only one that he called "the law of excluded middle", but that would be better called "law of dichotomy", to avoid the confusion with the previous formulation he gave of the law of excluded middle and because it better fits with what it really is. (WWR, V2, §9).

Schopenhauer says that metalogical truths "were discovered long ago by induction" and that:

> it is by means of a kind of reflection which I am inclined to call Reason's self-examination, that we know that these judgments express the conditions of all thinking, and therefore have these conditions for their reason. For, by the fruitlessness of its endeavors to think in opposition to these laws, our Reason acknowledges them to be the conditions of all possible thinking: we then find out, that it is just as impossible to think in opposition to them, as it is to move the members of our body in a contrary direction to their joints. If it were possible for the subject to know itself, these laws would be known to us *immediately*, and we should not need to try experiments with them on objects, i.e. representations."(4PR §33)

Since the justification of these metalogical truths is an important feature, let us also quote another formulation by Schopenhauer, similar but slightly different, that can be found in his *Handwritten Manuscript*, corresponding to his lectures in Berlin in the 1820s:

> The reason for these judgments is the consciousness of reason that only according to these rules one can think. However, reason does not come to the

> realization of this directly, but only through a self-examination, through a reflection on what can be thought (not experienced) at all. In this way it recognizes that it tries in vain to think against those laws; e.g. it cannot think that a circle is triangular, or a piece of wood of being iron: thus it recognizes those laws as the conditions of the possibility of all thinking. It is thus the same as we learn about the movements possible to the body (just as we learn about the properties of every other object) with the help of experiments. If the subject could recognize itself (what is, however, impossible), we would recognize those laws directly and not only with the help of experiments on objects, i.e. representations. (HDW, 1913, p.268)[16]

What is not clear and not detailed by Schopenhauer is how these four metalogical truths precisely emerged. The situation of the PRS of knowing seems a bit different from the three other metalogical truths. Considering the principle of non-contradiction, it rather seems that this principle was formulated by induction, not in the sense presented by Schopenhauer, i.e. that we cannot reason in a different way, but because everything in nature was seen as based on dichotomy (cf. the Pythagorean table of opposite). Then this natural phenomenon was transformed into an artificial device, classical negation, which became the main tool to develop reasoning. But we can indeed experiment without much problem other tools (see [28]).

Considering the downstream aspect, we can compare Schopenhauer with Aristotle. There are three different aspects of Aristotle's logic which are quite independent (in parentheses, their location in Aristotle's corpus):

- syllogistic, which is a system with rules describing and/or prescribing how to rightly reason (*Prior and Posterior Analytics*)
- criticism and description of false ways of reasoning (*Sophistical Refutations*)
- presentation and defense of the principle of non-contradiction (*Metaphysics*).

The relation between syllogistic and the principle of non-contradiction is clear neither with Aristotle, nor with Schopenhauer. In both cases the justification of the principle of non-contradiction is not based on any syllogistic argument and on the other hand syllogistic also does not seem to depend on this principle. It has even been argued that the rejection of the principle of non-contradiction is compatible with Aristotle's syllogistic (see [55]). However in the theory of the square of opposition the notion of contradiction is used to classify and organize the four kinds of propositions that are used in syllogistic. This would be a reason to consider the principle of non-contradiction as a metalogical principle in Aristotelean logic. But if we consider the square of opposition as part of the metatheory of syllogistic, it encompasses also two other notions of opposition (contrariety and subcontrariety) as well as subalternation.

Schopenhauer did not present a new logical system. He supports syllogistic and try to improve it, in particular by use of diagrams, further developing the works of Gottfried Ploucquet (17161790), Jean-Henri Lambert (1728-1777) and

[16] English translation courtesy of Jens Lemanski. No English translation of this *Handwritten Manuscript* has yet been published.

Euler, that he knew (WWR1, §9). He points out and praises the fundamental character of syllogism (corresponding to the cut phenomenon) that he describes as creative. He does that with two metaphors, one chemical, the other electrical:

> From *one* proposition there cannot result more than what is already to be found therein, that is to say, more than it itself states for the exhaustive comprehension of this meaning. But from *two* propositions, if they are syllogistically connected to premisses, more can result than is to be found in each of them taken separately; just as a body that is a chemical compound displays properties that do not belong to any of its constituent elements considered separately. On this rests the value of syllogisms. (PPA V2 §24)

> The voltaic pile may be regarded as a sensible image of the syllogism. Its point of indifference, at the centre, represents the middle, which holds together the two premisses, and by virtue of which they have the power of yielding a conclusion. The two different conceptions, on the other hand, which are really what is to be compared, are represented by the two opposite poles of the pile. Only because these are brought together by means of their two conducting wires, which represent the copulas of the two judgments, is the spark emitted upon their contactthe new light of the conclusion. (WWR §10)

But Schopenhauer claims: "we no more need logic to avoid false reasoning than we need its rules to help us reason correctly; and even the most learned logician completely puts it aside when actually thinking." (WWR §9) He makes a comparison with the two other corners of the basic triangle of the pyramid of philosophy made of truth, goodness and beauty, saying:

> We must yet remember that no one ever became an artist by the study of aesthetics; that a noble character was never by the study of ethics; and just as little do we need to know logic in order to avoid being misled by fallacies ... We do not have to burden our memory with all the rules, since logic can only be of theoretical interest and never of practical use for philosophy. It may be said that logic is to rational thought as the figured bass is to music, or, more loosely, as ethics is to virtue or aesthetics to art; but it should be borne in mind that no one has ever become an artist by studying aesthetics or achieved nobility of character by studying ethics, that people composed music both beautifully and correctly long before Rameau and that we do not need to have mastered the system of figured bass to recognize dissonance. In just the same way, we do not need to know logic to avoid being deceived by sophisms. (WWR1 §9)

Nevertheless Schopenhauer also has described 38 stratagems (in German: Kunstgriffe) which can be compared with the 13th fallacies described by Aristotle in his famous *Sophistical Refutations* (Fig. 10). It is part of an essay written in the 1830s which was not concluded during Schopenhauer's lifetime. It was published only after his death, sometimes presented in an ambiguous way with invented controversial titles or subtitles, such as *The Art of Being Right - 38th ways to win when you are defect*. In §26 of Volume 2 of PPA, entitled *On Logic and Dialectic*, Schopenhauer talks about this essay emphasizing the distinction between the form and matter of the sophisms:

> The tricks, dodges, and chicanery, to which they resort in order to be right in the end, are so numerous and manifold and recur so regularly that some years ago I made them the subject of my own reflection and directed my attention to their purely formal element after I had perceived that, however varied the subjects of discussion and the person taking part therein, the same identical tricks and dodges always came back and were very easily to recognize. This led me at this time to the idea of clearly separating the merely formal part of these tricks and dodges from the material and of displaying it, so to speak, as a neat anatomical specimen. I therefore collected all the dishonest tricks so frequently occurring in arguments and clearly presented each of them in its characteristic setting, illustrated by examples and given a name of its own. Finally, I added the means to be used against them, as a kind of guard against their thrusts; and from this was developed a formal *Eristical Dialectic*. (PPA, V2, §26)

Due to this comment the essay was posthumously baptised in German *Eristische Dialektik* and in English *Controversial Dialectic* or *Eristical Dialectic* (for a recent study on this essay, see [86]).

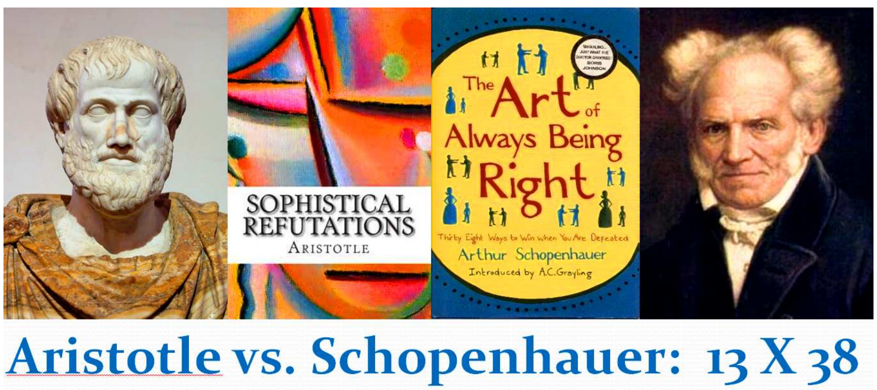


Fig.10 Schopenhauer won the sophistic game against Aristotle

But what prevails is Schopenhauer's critical view of the weak, not to say null, utility of the practical aspect of logic, as a tool for reasoning rightly and recognizing wrong reasoning. This leads him to make the following consideration:

> The teaching of logic should not take the form so much of a science oriented towards practice, and should not merely set down unembellished rules for the correct conversion of judgments and inferences etc.; instead it should be directed towards making known the essence of reason and concepts, and towards a detailed consideration of the principle of sufficient reason of knowing. After all, logic is merely a paraphrase of this principle, and indeed only for cases in which the ground for a judgment's truth is neither empirical nor metaphysical but rather logical or metalogical. In addition to the principle of sufficient reason of knowing, we must introduce three more fundamental laws of thought or judgements of metalogical truth that are just as closely related; the whole technique of reason emerges little by little from these. (WWR1 §9)

If we reduce a course of logic to Schopenhauer's Theory of the Metalogical it would be a quite short course, because he does not say much about the metalogical truths. After having located the metalogical truths in his system, as one of the 4 kinds of truths, itself part of the 4th root of the PRS, and stated that they are 4, Schopenhauer does not go much further. As he himself writes: "I attribute metalogical truth to these laws because they come purely from reason and are not to be explained any further." (WWR Appendix on Kant). No comments!

On the other hand in the above quote Schopenhauer suggests to include in the teaching of logic, the knowledge of the *essence of reason and concepts* that he puts side by side with the PRS of knowing and other metalogical truths. It seems reasonable to indeed consider that Schopenhauer's Theory of the Metalogical does not reduce to the metalogical truths but includes his ideas on reason and concepts which are directly connected with them. This is what we will explore in the next section.

2.3. The Femininity and Triviality of Metalogical Truths

The essence of reason is not necessarily something easy to catch. Especially if we consider that reason is the essence of human beings, a basic idea promoted by the Ancient Greeks that Schopenhauer fully embraces despite his fondness for dogs, music and the Buddha. He considers this idea as truly universal:

> The unanimous view of every age and people is that these various and far-reaching manifestations all spring from a common principle, from a special mental power that distinguishes humans from animals and that is called *Vernunft*, *Loogos*, *Ratio*. (WWR1 §8)

He starts by saying that reason is easily recognized and qualified by everybody:

> Everyone also knows very well how to recognize the manifestations of this faculty, and can tell what is rational and what is irrational; everyone can tell where reason emerges in contrast to the other human capacities and characteristics.

and that even philosophers agree about it:

> The philosophers of all ages also generally agree with this common knowledge of reason, and in addition emphasize several of its especially important manifestations: mastery of affects and passions, the ability to make inferences and to lay down universal principles, even those that can be ascertained prior to any experience, etc.

but Schopenhauer adds:

> However, all their explanations of the true essence of reason are wavering, vaguely delineated, long-winded, and lack both unity and focus. (WWR1 §8)

Let us see if the poodle philosopher himself has a clear and distinct explanation of the essence of reason.

Schopenhauer writes: "Reason is of a feminine nature: it can give only after it has received. On its own, it possesses nothing but the empty forms of its own operation. Completely pure rational cognition gives us in fact only four things, the very metalogical truths." (WWR, §10) From this we can infer that metalogical truths are the feminine structure of our thought.

We can illustrate Schopenhauer's metaphor with a 1878 painting by Charles Louis Müller (1815-1892), entitled "La fête de la Raison dans Notre-Dame de Paris le 10 novembre 1793" (Fig.11).

Fig.11 Chaumette symbolizing Reason at Notre-Dame in 1793

We have chosen this painting because Reason is herein represented, not to say advertised, by a woman, but also because this celebration of reason took place in the most famous church of Paris (Notre-Dame) and was supposed to replace the erroneous religious cult. During this ceremony, the girl, named Chaumette Momoro, made the following declamation:

> Vous l'avez vu, citoyens législateurs, le fanatisme a lâché prise et a abandonné la place qu'il occupait à la Raison, à la justice, à la vérité; ses yeux louches n'ont pu soutenir l'éclat de la lumière, il s'est enfui. Nous nous sommes emparés des temples qu'il nous abandonnait, nous les avons régénérés. Aujourd'hui tout le peuple de Paris s'est transporté sous les voûtes gothiques, frappées si longtemps de la voix de l'erreur, et qui, pour la première fois, ont retenté du cri de la vérité. ([1], p.301)

Schopenhauer, who claimed that "a man cannot serve two masters, so it is either reason or the scriptures" (PPA, V2, Ch15), could have been a follower of Chaumette (in 1793 however he was only 5 years old). The choice of this painting is also to emphasize the problematic rationalism of Schopenhauer, grounded on emptiness.

The feministic view of reason promoted by Schopenhauer is compatible with Aristotle's views on logic. It fits well with the Stagirite's hylemorphism, which at the logical level corresponds to the distinction between the form of a reasoning and its matter. The form of reasoning is described by Aristotle with his syllogistic figures and the matter is the possible interpretations of the subject and the predicate which can be anything fitting within this dual categorization (Socrates and other individuals being excluded).[17] As we have emphasized in previous papers ([16], [22]) this formal character of logic is one aspect of Aristotelean logic that is still predominating in modern logic. What has changed is the form of the form not the essential formal nature of logic. Nobody indeed seems shocked by the use of the expression "formal logic" in modern logic, expression due to Kant who famously claimed that Aristotle's logic will never changed (Preface of the 2nd edition of the *Critique of Pure Reason*). And in fact its hylemorphic character has survived.

In case of Schopenhauer we can also see a difference on the form with Aristotle. He considers that the form is characterized by metalogical truths, not syllogisms. According to him metalogic truths are "formal conditions of thought". But on the other hand Schopenhauer says:

> The essence of thought proper, i.e. of judgment and inference, can be presented by combining conceptual spheres according to the spatial schema described above, and all the rules of judgment and inference can be derived from this schema by construction. The only practical use that can be made of logic is to prove that an opponent in debate is using intentional sophistries (not making genuine logical mistakes) by pointing out their technical names. (WWR1 §9)

The articulation between this combinatorial essence of thought and the feminine metalogical one is not explicitly explained by Schopenhauer. We can see that all these essences are formal. We can argue that the combinatorial essence is produced or justified by the metalogical one (our paper "Opposition and order" goes in that direction [23]). Schopenhauer was not able to properly explain the phenomenon as he himself recognizes: "I am unable to say what the ultimate basis is for this exact analogy between the relations of concepts and those of spatial figures. But it is in any event a fortunate circumstance for logic that the very possibility of all conceptual relationships can, in the following way, be presented intuitively and *a priori* by means of such figures ... All combinations of concepts may be reduced to these cases." (WWR1 §9) This is here another example of use of spatial devices to reason about something else than geometrical space. And this is especially important, because it is an application of spatial devices to describe and explain reasoning.

The square of opposition is another geometrical figure which is very famous on the history of logic. This square is at a deep metalogical level if we consider that it gives an explanation of what are the different categorical propositions by classifying them, using in particular the notion of contradiction to do that, and that moreover it explains what contradiction is by on the one hand distinguishing it from other oppositions and on the other hand showing how it works, giving an example of application. By contrast, spheres of concepts are devices to describe

[17] About the distinction between form and matter in syllogistic, see [32], [33], [33].

and/or practice reasoning, although they also have a theoretical systematic aspect emphasized by Schopenhauer, explaining all the possibilities, providing the general picture. The two kinds of figures, square and circle, can be mixed together in the diagram presented in Fig.12, an improvement of the one by Tilman Plesk, which is presently the main top illustration on the entry about the square on Wikipedia. We have replaced the inside square by a colored square, where red represents contradiction, blue contrariety, green subcontrariety and the black arrows, subalternation, using hence colors additionally to spatial representation, putting in activity another one of our senses (about the square and colors see [21] and [66], and for alternatives of Fig.12, see [4]).

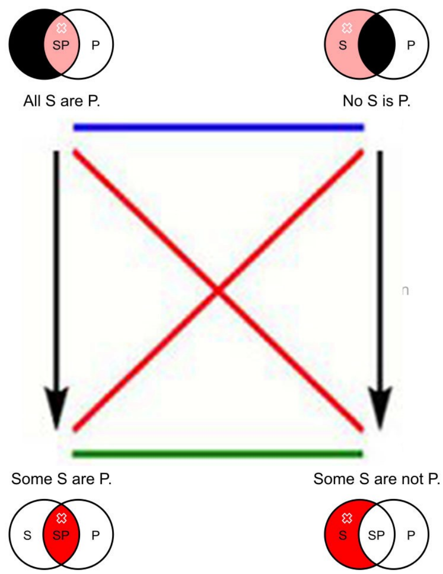


Fig.12 Circling the Square of Opposition

Schopenhauer also pretends to explain the relation between reason, conceptualization and understanding, but his explanation is rather strange and complicated. He says: "Reason has only *one* function: the formation of concepts, and all the phenomena mentioned above can be very easily and in fact trivially explained on the basis on this simple function: it is what distinguishes the life of humans from that of animals; and everything that has been, at any time or place, described as rational or irrational points to the application or non-application of this function." (WWR1, §8 p.62)

Concepts are abstracts but they are not abstraction of reality, they are more like reflects of the reality of phenomena. They are representations of representations. The basic representations are intuitive representations ruled by the law of causality (PRS of becoming) which is connected to understanding (cf. WWR1 §8 and 4RP §26 and §27). Schopenhauer reinforces his feminal metaphor by contrasting the femininity of reason to the masculinity of understanding: "... its nature is feminine; it only conceives, but does not generate. It is not by mere chance that the Reason is feminine in all Latin, as well as Teutonic, languages ; whereas the Understanding is in variably masculine." (4RS, pp.136-137d). This metaphor which is duplicated in a Sun-Moon metaphor, the intuitive representation being equated to the Sun, and the concepts to the Moon (Fig.13): "As if from the direct light of the sun into the borrowed reflection of the moon, we now pass from immediate, intuitive representation (which presents only itself and is its own warrant) into reflection, the abstract, discursive concepts of reason (which derive their entire content only from and in relation to this this intuitive cognition). (WWR1, §8).

Fig.13 The Moon: the Kingdom of Logic
with Concepts of Reason and Boole's Crater

What is difficult to understand in the philosophy of Schopenhauer is not only the articulation of the PRS of becoming (corresponding to understanding) with the PRS of knowing (corresponding to reasoning) but also the articulation between these two and the PRS of being (corresponding to mathematics). For example on the basis on these three principles how Schopenhauer would explain how work a physical theory making use of mathematics like Newton's theory of gravitation or Einstein's theory of relativity based on Non-Euclidean geometry? A central point of Schopenhauer's theory on which he himself insists is that there are not four different separated PRS but one PRS with four roots as indicated by the very title of his essay. That's nice but it is not completely clear how everything is articulated especially considering some claims of Schopenhauer which look a bit paradoxical, at least as they are phrased, such as "understanding, considered in itself, is unreasonable" (WWR1, §6), or his "phenomenal" claim according to which science has nothing to do with the inner essence of the world (WWR1, §7).

3. Formulation, Axiomatization, Interaction, Reflection

Schopenhauer had the idea that the fundamental basis of reasoning that he claimed was fairly described by what he characterized as the four metalogic truths was predetermined and fixed and therefore would never change. We will examine in the third part of this paper if this idea makes sense in the light of the development and evolution of modern logic, in particular by making the distinction between reasoning, its formulation and description. For conducting this analysis we will take advantage of the clarification about modern Metalogic made in the first part of our paper.

3.1. Reformulations, Semiotical Changes and Mathematical Interaction

According to *Encyclopaedia Britannica*, the *laws of thought* are "traditionally, the three fundamental laws of logic: (1) the law of contradiction, (2) the law of excluded middle (or third), and (3) the principle of identity." [46] Schopenhauer also calls them laws of thought, but he additionally characterizes them as "metalogical truths". And the original contribution of Schopenhauer is also to have considered, as we have seen, a fourth law, the PRS of knowing, according to which: "truth is the ratio of a judgment to something out of it". He uses the same name for these laws but did not formulate them in the same way as in *Encyclopaedia Britannica*. Here is his formulations:

- (identity) A subject is equal to the sum total of its predicates, or $a = a$.
- (contradiction) No predicate can be attributed and denied to a subject at the same time, or $a = -a = o$.
- (excluded middle) One of two opposite, contradictory predicates, must belong to every subject.

This is rather unsatisfactory. From a modern point of view this is rather weird, not to say false. But let us note that in *Encyclopaedia Britannica* these laws are also formulated in a unsatisfactory way: (1) and (2) are expressed in the language of modern propositional logic and (3) of first-order logic. Considering that the entry is about the *traditional* laws of thought, this is an anachronism. And we have a disparity, because (1) and (2) are expressed in propositional logic and (3) in first-order logic, moreover (1) and (2) are called "laws" and (3) a "principle".

We will not here develop a critical analysis of Schopenhauer's formulations because on the one hand this would require a better understanding based on a historical and philological research, comparing formulations of these laws by other authors of the period (in the line of the work of Anna-Sophie Heinemann included in the present volume [58]), and on the other hand this is not so important for our present discussion.

The first important point is that these formulations are not proper original inventions of Schopenhauer, it is at best a reformulation of something which was in the air at his time. Schopenhauer indeed does not emphasize or claim any personal contribution. Before presenting the above formulations, he writes: "There are only four metalogically true judgments of this sort, which were discovered long ago by induction, and called the laws of all thinking; although entire uniformity of opinion as to their expression and even as to their number has not yet been arrived at, whereas all agree perfectly as to what they are on the whole meant to indicate." (PRS 33) Writing this he does not even consider that the fourth metalogical truth is due to himself, as it is indeed the case, as well as the idea to put it at the same level as the three other metalogical truths.

The second important point is that Schopenhauer was not interested to question or furthermore investigate these formulations. His position can be understood on the basis that according to him, although we don't have a direct access to them (we know them only by self-reflection), there are obvious, they don't need further explanation, things cannot be otherwise, it is like the way we use our foots for walking. Let's go on and ahead with this walking metaphor:

- (WALK-1) We are walking in a certain way, we see how it is by practicing it.
- (WALK-2) It is not possible to better walk, to walk in a different manner.
- (WALK-3) It is not useful to further describe how we are walking.

Schopenhauer's position is in accordance with these three points. This also the case up to a certain point of the positions of René Descartes (1596-1650) and Blaise Pascal (1623-1662). But both French philosophers think that syllogistic is not only useless but also misleading – Schopenhauer is not so critical – and they both present new methodologies, that we have summarized in two tables in [17]. Descartes's methodology is very general and quite far from any logical principles or systems (although exhaustion can be viewed as an extended version of the principle of excluded middle). Pascal's methodology is the promotion of the axiomatic method, it has strongly inspired Tarski and has some connections with our present discussion on Metalogic, we will deal with this in the next section. Schopenhauer's position is much more conservative. Nevertheless he has two original contributions we have talked about in the second part of our paper: to reduce the three traditional laws of thought to only one, to consider five possible basic positions between spheres of concepts. His ideas are interesting but not presented in a very satisfactory way and moreover he does not develop them much.

Kant had the idea that Aristotle's science of logic was perfect. Maybe he gave more value to it than Descartes, Pascal and Schopenhauer. Kant famously claimed in the preface of the second edition of the *Critique of Pure Reason* that the science of logic is firmly and definitively established, therefore a dead science. Schopenhauer's position is in the same vein as the one of Kant, despite some light differences.

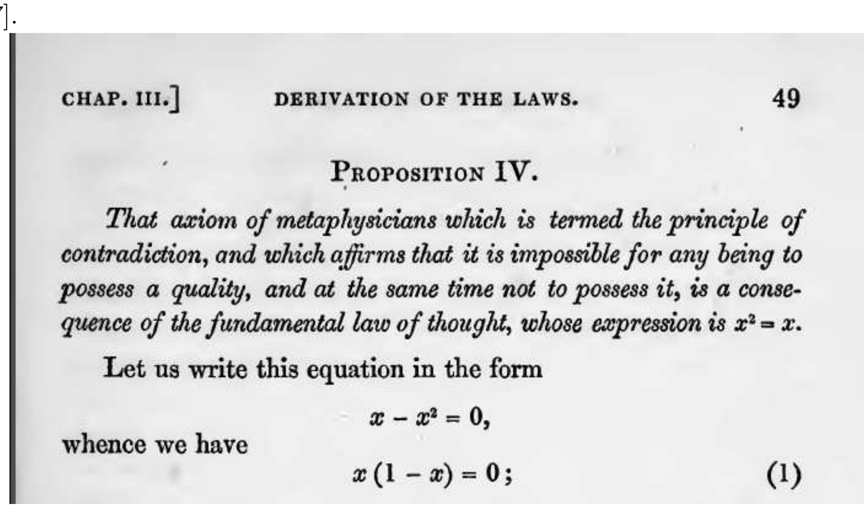
Ironically, few years after Kant's morbid declaration (1787), was born a man known as George Boole, who developed a line of work, when Schopenhauer was still alive, that revolutionized the science of logic and from which modern logic arose. It is worth to stress however that Boole was not a revolutionary by birth, nature or behavior (this was more the case of his wife, a forerunner of homeopathy, who by experimenting it on her husband caused his premature death, throwing cold water on him, after he went back home strongly wetted by a heavy Irish rain). So how to explain that Boole changed the history of logic?

We can understand that through an important distinction. Modern logic has challenged the traditional laws of thought in two different ways, not to be confused:

- The traditional laws of thought are not necessarily fundamental laws.
- The traditional laws of thought do no not always hold.

We will discuss the first point here and the second in the next section. The first point means that a law, like the law of non-contradiction, can be derived from some more fundamental laws (but still is valid). This point was made by Boole. This is one of the reasons he can be considered as the father of modern logic. And this is related to one of the characteristics of the new methodology he promoted, i.e. to use mathematical tools and symbolisms to develop logic, which is also typical of modern logic.

Boole considered that the fundamental law of thought is $x^2 = x$. He claimed that in his famous 1954 book the *Laws of Thought*, a claim supported by a "proof" that it is possible to derive from it the law of contradiction using the symbolism he is promoting (cf. Fig.14). We have examined this point in details in a recent paper entitled "Is the principle of non-contradiction a consequence of $x^2 = x$?" [27].



CHAP. III.] DERIVATION OF THE LAWS. 49

PROPOSITION IV.

That axiom of metaphysicians which is termed the principle of contradiction, and which affirms that it is impossible for any being to possess a quality, and at the same time not to possess it, is a consequence of the fundamental law of thought, whose expression is $x^2 = x$.

Let us write this equation in the form

$$x - x^2 = 0,$$

whence we have

$$x(1 - x) = 0; \qquad (1)$$

Fig.14 Boole's symbolic proof that the principle of contradition is derivable from $x^2 = x$, the fundamental law of thought for him

Few years later (1880) Peirce showed that it is possible to derive/define all the 16 connectives of classical propositional logic with only one. He did that in a semantical way, using a method similar to what is now presented as the bivalent semantics of propositional logic. Later on Jean Nicod (1893-1924) showed that it is possible to axiomatize classical propositional logic with this sole connective. Wajsberg gave another version of such axiomatization and Henry Sheffer (1882-1964) independently rediscovered this connective. Whithead, Russell, Wittgenstein knew this result and thought it was important, and in fact it is (for details about his connective see [81]). Not only the principle of contradiction and excluded middle can be derived from this connective, but also all other principles governing classical connectives, in particular the one corresponding to what Boole considered as the fundamental law of thought. Peirce therefore went a step ahead of Boole.

But what is important is that in the cases both of Boole and Peirce there is a crucial semiotical change in the very formulation of logic (and this is also the case later on with a third famous father of modern logic, Frege). Boole was much influenced for that by the British school of *Symbolic Algebra*. He did perform the semiotical change of considering operations on signs rather than on their values due to this school but made a fundamental new step by considering algebraic operations on signs having values other than numbers or quantities (see [44], [45], [104], [129]). Peirce's contributions to semiotic is well-known. He indeed is considered as the father of semiotics and had the idea that logic is part of semiotics.

Many people have the tendency to reduce the development and emergence of modern logic to a phenomenon of *formalization* of logic. But as we have pointed out in [16] the expression "formal logic" is highly ambiguous having 5 different meanings. So it is better to say that the changes who led to modern logic were due to new *formulations*, based on semiotical changes. And to point out that these semiotical changes cannot be characterized or reduced to a "mathematization of logic". Modern logic was inspired by mathematics but also changed mathematics, what we have is a real interaction.

Results like those of Boole and Peirce can be considered as metatheorems, part of Metalogic, but all the semiotical aspect of their work leading to a new conception of logic also can be considered as part of Metalogic. Schopenhauer's Theory of the Metalogical is far from all this but at the time it is interesting to see that he is lightly touching this dimension, on the one hand by promoting spherical diagrams, which are in the spirit of Euler and partly resemble the so-called "Venn diagrams", developed by John Venn, (to whom is attributed the expression "symbolic logic" [122] and who is considered as an important figure of modern logic), on the other hand by using some mathematical symbols to express the law of identity and contradiction. He is however not doing that for the two other metalogical truths: the law of exluded middle and the PRS of knowing. Although the first three metalogical truths discussed by Schopenhauer have been reformulated and relocated (they are not necessarily at fundamental first positions) in modern logic, they still are there at the logical or metalogical level. On the other hand the PRS of knowing, an original idea of Schopenhauer, has completely disappeared. Heinrich Scholz (1884-1956), a good friend of Łukasiewicz, wrote a book on the history of logic [98] where he claimed that this is because it cannot be formalized. Against this view Newton da Costa presented a formalization of it using modal propositional logic with quantification on propositions (see [7], [9]). This idea has not yet been systematically developed. Doing so could lead to an interesting new logical theory inspired by Schopenhauer's Theory of the Metalogical.

The criticism and rejection of the traditional laws of thought were not initiated by Boole, Peirce or Frege. From this point of view we can say that they were not fundamentally against Schopenhauer's metalogic truths. On the other hand if we consider that semiotics is a fundamental part of Metalogic, we can say that their metalogical views are quite different from those of Schopenhauer.

3.2. The Modern Axiomatic Methodology

In the previous section we have seen how the law of non-contradiction was relativized by Boole and Peirce, being derivable from other laws, this being done by the reformulation of the basic logical framework. In this section we will see another point: the rejection of the traditional laws of logic, in particular the very law of non-contradiction. One of the first to perform this rejection was the Russian logician Nicolai Vasiliev. We already talked about him in section 1.2. pointing out he was using the word "Metalogic". His relativization has to be understood through his conception of Metalogic but also through his promotion of the axiomatic method. When we are talking about the axiomatic method, we are talking about the new axiomatic method, which was in particular promoted from Kazan, the city of Vasiliev's family (his father was a friend of Nikolai Lobatchevski). Vasiliev considered that the principle of non-contradiction can be treated as the parallel postulate:

> Non-Euclidean geometry is a geometry without the 5th postulate, [that is] without the so-called axiom of parallels. Non-Aristotelian logic is a logic without the law of contradiction. It is worth mentioning here that it was precisely non-Euclidean geometry that has served us as a model for the construction of non-Aristotelian logic.([120], p.128).

According to the modern axiomatic methodology, a very important tool of the modern world (Fig.15), axioms are relative in two complementary and non exclusive senses:

(A) An axiom can be replaced by another one.
(B) An axiom is not considered as an absolute truth.

(A) is quite specific of modern axiomatic and was emphasized by Tarski (see [111] and chapter 6 of [112]) . It is related to, but not only, the formulation of axioms with different primitive terms, primitive terms that are also therefore not absolutely primitive. A theory can be axiomatized in different ways. A Boolean algebra can for example be seen as an idempotent ring or as a distributive complemented lattice. And a given axiom can have many different equivalent formulations, a typical example is the one of the axiom of choice.

(B) is not completely new. Plato had the idea that mathematics was based on hypotheses rather that on absolute truths. The search for truth was for him the task of philosophy, a therefore higher science (cf. Book 6 of *Republic*).

What is the most important is that modern axiomatic was applied to logic, and this led to the relativization of logical axioms not only in the sense of (1) but also of (2). There are different logical systems starting with different axioms leading to different theorems. The study of the different logical systems is part of

Metalogic in the Polish sense, as we have explained in section 1.2. To play with axioms of logic was a favorite game in the 1920s. In Poland they liked the idea to reduce everything to one axiom. An other idea was to develop independent axiomatizations, in the sense that in an axiomatic system one axiom cannot be derived from the other ones. Paul Bernays in his PhD ([5], [6]) showed that the system of axioms for propositional logic in the *Principia Mathematica* was no independent and provided an independent one, showing its independence using three-valued matrices. Independence is a typical metalogical concept or/and result. The use of many-valued matrices to prove such a metatheorem is part of Metalogic, in a more essential way that the use of such a device to develop a logical system, because a logical matrix, or set of logical matrices, can be seen itself as a logical system.

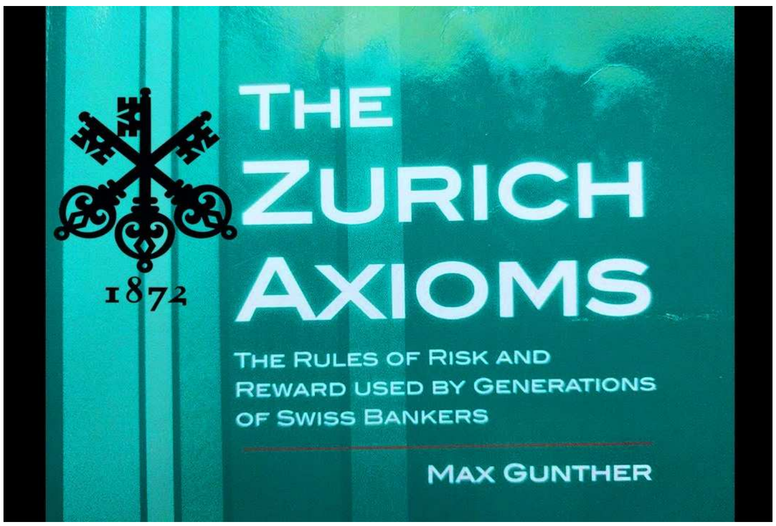


Fig.15 The Axiomatic Method: a Winning Strategy

If we consider an axiomatic theory, let's say Peano Arithmetic (PA), and a theorem of this theory, let say the infinity of prime numbers (IP), according to Schopenhauer's terminology, this is a *logical truth*, because its reason consists in other judgments, ultimately PA axioms. Although this fits with the spirit of modern logic, the language used is not the same. In modern logic it is said that a theorem of PA logically follows from the axioms of PA, but not that it is a logical truth.

In modern logic the expression "logical truth" is attributed to truths which are not depending on non-logical axioms, such as the axioms of PA. We say that a proposition is a *logical truth* if it is true in virtue of logic itself. Alternatively it is synonymously said that such a proposition is *logically valid* or that it is a *tautology*.

The infinity of prime numbers is not a logical truth in the sense of modern logic, but the formula PA' $\rightarrow$ IP is.[18] This fact can be used to argue that all results of mathematics are nothing else than tautologies or formal truths. This is a position defended in particular by philosophers who know very few about mathematics, who don't know what is the thrill of proving a theorem by being directly in touch with beautiful mathematical objects, not to say creatures. Working mathematicians who are living for and from such thrills don't support a tautological view of mathematics (although there are some exceptions such as Saunders MacLane, see [79]). They would certainly be more sympathetic to Schopenhauer's philosophy of mathematics. In fact despite the fact that Arithmetic has finally been axiomatized after several thousands of year (by contrast to Geometry which was axiomatized right at the start), generally mathematicians working in number theory are not interested to show how their theorems can be derived from PA.

In an axiomatic system for a mathematical theory like Peano Arithmetic, the rules are supposed to be orchestrated (specified and described) by logic. Now if we have an axiomatic system for logic, is logic itself orchestrating the rules and how? Is this Metalogic?

If we consider a tautology such as $p \vee \neg p$, can we say that it is a metalogical truth in the sense of Schopenhauer? One could claim that, arguing it is a modern formulation of what Schopenhauer calls the law of excluded middle, one of his four metalogical truths. We can consider a Hilbert proof-theoretical system in which the formula $p \vee \neg p$ is an axiom. From this axiom and another axiom expressing the commutativity of disjunction, it is possible to prove that $\neg p \vee \ p$ is a theorem (not a metatheorem). We can go the other way round: start with $\neg p \vee \ p$ as an axiom and derive $p \vee \neg p$ as a theorem. This illustrated the point (A). In this example there is nothing dramatic because the two formulas are quite the same and both can be called "excluded middle". The situation is different if we derive $p \vee \neg p$ from $((\neg p \rightarrow q) \wedge (\neg p \rightarrow \neg q)) \rightarrow p$, a formula corresponding to the strong version of the reduction to the absurd. It is indeed possible to do so not using other axioms or rules for negation but only principles ruling implication, disjunction and conjunction. It would not make really sense to call this formula, then an axiom, the excluded middle. From it, it is also possible to deduce in fact various formulas that can be interpreted as formulations of the law of non-contradiction. As we have seen, Schopenhauer had a similar proposal, deriving the law of excluded middle from a more fundamental law from which he says the law of non-contradiction can also be derived. For Schopenhauer however this does not change the essential value of the law of excluded middle, it is still a metalogical truth.

From the viewpoint of modern logic, even if we agree that $p \vee \neg p$ is a formula having a real axiomatic value, not just a formula lost in the infinite jungle of all formulas, it would be a bit strange to call it a metalogical truth. The reason why is that it is awkward to apply the metaterminology to axioms of a logical

[18] PA' is here the conjunction of the propositions of a finite subtheory of PA, from which IP is a consequence.

systems, because they are part of the system in the same way as the theorems. The metaterminology is reserved for things about the system. For example dedicability, which is considered as a metatheorem. That's the reason why it does not make sense to talk about metaaxiom, unless we develop a full metatheory such as Tarski's consequence operator which can be seen as axiomatizing the properties of a consequence relation generated by a Hilbert system.

This is an important difference between Schopenhauer's Theory of Metalogic and the modern one, because Schopenhauer is not going that high as a theory of consequence. The second important difference is that in modern logic we have on the (B) side the rejection of the law of excluded middle, an easy game, which is in fact facilitated by the axiomatic method applied to logic but also to its metatheory. First of all we can construct a Hilbert proof-theoretical system in which $p \vee \neg p$ is neither an axiom, nor a theorem, the canoncial example being Heyting's system of intuitionistic logic. In this system in which $p \vee \neg p$ is not an axiom, to prove that it is also not a theorem is a metatheorem. This metatheorem can be proved in various ways in particular using logical matrices.

The intuitionistic system of logic can also be generated by a Gentzenian sequent system. What is surprising is that the standard system presented by Gentzen for intuitionistic logic has the same logical rules as the system for classical logic, in particular the same rules for negation. The difference is at the level of the structure of the system, not at the level of the structural rules, but at the level of external determinations: the sequents being not the same as the classical ones, having only one formula on the right. As we have described the situation in a previous paper (see the table *The Architecture of Sequent Systems* in [13]), in sequent systems the structural principles can be divided into internal ones (structural rules) and external ones. It would not make sense to both call them metalogical principles because there are at two different levels, and the meta prefix contains the idea of differentiation of levels. But for course all this corresponds to the field of Metalogic that we indeed prefer to call *Universal Logic* as emphasized in section 1.3. This change of terminology is also important to stress that Metalogic does not reduce to an axiomatic game, that the foundation of logic, if any, is much more conceptual and semiotical.

3.3. Multi-Level Analysis and Productive Self-Reflection

For Schopenhauer metalogical truths are not immediately and directly perceptible, nevertheless they are obvious. Louis Rougier (1889-1982), promoter of the Vienna Circle, in an interesting book with a beautiful poetic title *Les Paralogismes du Rationalisme* (*Paralogisms of Rationalism*) published in 1920 criticized rationalism based on some obvious truths like "the whole is bigger than the part", one of his favorite targets being Leibniz. These considerations and seeing himself later on the development of modern logic with many non-classical systems led him to a spectacular, not to say dramatic, claim: "Avec la découverte du caractère conventionnel et relatif de la Logique, l'esprit humain a brûlé sa dernière idole" (in English: With the discovery of the conventional and relative character of logic, human spirit has burnt his last idol) [96]. But, as we have said in a previous paper [19], at the end Rougier defends a wishy-washy scientism, contrasting with his smashy declaration.

Fig.16 Shall we burn Reason with her four Metalogical Truths?

More interesting was the behavior of one of the main leaders of the Lvov-Warsaw School, namely Jan Łukasiewicz. In the prehistory of this school (1910), he wrote a book in which he precisely analyzes and criticizes Aristotle's arguments supporting the principle of non-contradiction [74]. Such an approach was not motivated by an ideology according to which contradiction is the basis of everything but by a rational inquiry. In an appendix of this same book, Łukasiewicz presents the ideas of Ernst Schröder (1841-1902) about logic. According to Jan Woleński [132], this is the first presentation of "formal logic" in the circle that will become one of the most important schools of modern logic. Later on Łukasiewicz was led to construct a formal system of logic not rejecting the principle of non-contradiction (this was done in Poland much later – 1948 – by Staniłsaw Jaśkowski (1906-1965)[65], for reasons having nothing to do with Łukasiewicz's book), but rejecting the principle of excluded middle [75]. And this was not done in a non-Aristotelian perspective, on the contrary, this was done in view of supporting Aristotle's views on future contingents. Łukasiewicz's logic is both a three-valued logic and a modal logic.

As we have seen in section 1.2., Metalogic *stricto sensu* is the study of some logical systems. But we can consider that the philosophical analysis of basic laws of logic is part of Metalogic *lato sensu*, as well as the creation of new logical systems generated by this analysis, systems developed by a methodology which itself is part of Metalogic *stricto sensu*, whether it is the use of logical matrices, sequent systems or possible worlds.

Łukasiewicz's three-valued logic is not against Aristotle, but it goes a step further by on the one hand giving a better *understanding* of contingency and the possibility to go beyond the truth-falsity dichotomy, on the other hand providing *techniques* with some useful applications.

Now let us examine the law of non-contradiction, also considered by Schopenhauer as a metalogical truth. If we consider the theory of the square of opposition, we could say that Aristotle was not absolutely defending this law, since in the square, among the three notions of opposition, there is subcontrariety, according to which two propositions can be true together and opposed. But that would be an anachronical and false view, because Aristotle explicitly says that he does not consider subcontrariety as an opposition (cf. *Prior Analytics*, 63b21-30). The square of opposition with 3 oppositions was firmly established only later on, in particular by Apuleius and Boethius (see [37]). Nevertheless Aristotle introduced the basic distinction which led to the square, the distinction between contrariety and contradiction. The introduction of contrariety next to contradiction according to which two propositions can be false together, can be seen at the same time as a rejection of the excluded middle and as a relativization of the notion of opposition, not reducible anymore to contradiction. This indeed can even be interpreted as a relativization of the notion of contradiction and the related principle of non-contradiction.

As we have pointed out in a previous paper [14], it is possible to establish a correspondence between the 3 notions of opposition of the square of opposition, contradiction, contrariety and subcontariery and the 3 kinds of negation, respectively, classical, paracomplete and paraconsistent negations. This does not mean that all aspects of negation are already inside the square, but the square is a general picture.

The understanding of the law of non-contradiction can and has been developed in different ways in modern logic. There are various formulations both syntactical and semantical. And what is very interesting is the study of the relation between this law and other properties of negation. It is possible to put all the properties of classical negation into one axiom, the strong version of the reduction to the absurd, from which everything can spring, not only the law of non-contradiction and excluded middle, but also the *ex-falso sequitur quod libet*, all versions of contraposition, etc. This is nice, but what also is nice is the complete deconstruction of this very single axiom in many pieces and the relations between these different pieces (see [10]). To do that we don't have to take a position, to believe or not that the law of non-contradiction is absolutely true. It is indeed better to carry on these metalogical investigations in a neutral and objective way.

And it is better to consider that these investigations are part of "Universal Logic" rather than "Metalogic". First because the properties of negation are at different levels: a logical level, like a property of negation expressed by a formula such as $\neg(p \wedge \neg p)$, or at the metalogical level, like the replacement theorem, according which for example $\neg(p \wedge q)$ is logically equivalent to $\neg(q \wedge p)$. It is a bit confusing to call Metalogic the study of the relation between logical and metalogical properties. To call it "Metametalogic" would on the one hand not be very nice and on the other hand would not solve the problem, because we may need to go to a further meta-level. The second reason to choose "Universal Logic" is that one of the central ideas beyond Universal Logic is a systematic comparative study of all logical systems, the examination of the different properties of negation being a natural part to this study. And a third reason is that Universal Logic does not reduce to mathematical or/and formal studies of the properties of logical systems (and/or logical operators such as negation), there is also a philosophical dimension. For example in the case of negation the idea is to simultaneously study the technical properties of negation, their interpretations and meanings. We can then really see if the law of non-contradiction makes sense or not and if it is possible to reason in a coherent way without it or with only part of it (see [28]).

A logical system in which there is a negation not obeying the full law of non-contradiction is called a "Paraconsistent Logic" and such a negation is called a "Paraconsistent Negation". Newton da Costa (1929-20??), who chose this terminology [38], started to work on this topic motivated by Russell's paradox, according to which the *principle of abstraction* leads to a contradiction. The principle of abstraction states that every property determines a set. It can be seen as an axiom or a fundamental law of thought. This was not done by Schopenhauer or other "traditional" logicians, who did not think of it as a principle but rather as a mechanism of conceptualization. The obviousness of this principle of abstraction can be seen as higher as the one of the principle of non-contradiction. One may then want to reject the principle of non-contradiction if this allows to save the principle of abstraction. Unfortunately this does not work in an easy and simple way due to Curry's paradox [41], which is a version of Russell's paradox using only some basic properties of implication.

In modern logic, logical systems without negation have been studied, the most well-known being *positive propositional logic*. In some sense we can say that these systems reject the law of non-contradiction, since they are not even involving negation. And we can also say that in the case of a metalogical system like Tarski's original theory of consequence in which at the first stage no connectives at all are involved.

Someone may say: that's very fine! but which logic are you using to do all that? Certainly all these investigations cannot be packed in one big logical system. They are not carried on in one system. We can consider various systems reflecting them and reflect about these systems, *ad infinitum...*

We can agree with Schopenhauer that we don't immediately know/perceive the laws of reason, and we can even go further saying that even when our reason is put in action they don't fully unveiled. Someone may think that we are more pessimistic than the king of pessimism. In some sense it is true, but we can see also a beauty in that: the unveiling is possible and pleasant, and this is an infinite pleasure, because it never ends, there is no final understanding.

Would it be interesting to face the very essence of reason depicted as the metalogical truths formulated by Schopenhauer or in another way? If we have seen it, so what? Or: what then can we do?

Someone may look at her face in a mirror but that would be a mistake to think that she then knows who she is (Fig.17). Self-knowledge is not that easy. This face in the mirror is just one of her aspect. And it is also not by seeing her whole body naked in the mirror that she will reach complete full-fledged self-knowledge of herself.

Fig.17 To See and Not to See: that's the Question!

We don't only have to unveil, we also need to act or, better, to interact. In the case of logic, we have to reason about reasoning.[19] By doing so we have a better understanding of what reasoning is and we further develop reasoning, getting higher.

[19] Roy Cook says: "Metalogic can be captured, loosely, by the slogan *reasoning about reasoning*, we agree with him but we don't reduce reasoning about reasoning to metalogic as he describes it, i.e. the "mathematical study of formal systems that are intended to capture correct reasoning." [36], p.188.

Acknowledgements and Memories

I started to be interested by Schopenhauer when I was 20 years old, in particular by reading two books by Clément Rosset (1939-2018): *Schopenhauer, philosophe de l'absurde* [92] and *L'Esthétique de Schopenhauer* [93]. I started then to read most of the works of Schopenhauer and up to know this is the philosopher I have read the most and who I think is one of the greatest philosophers of all time. I have been interested in all the aspects of his philosophy (religion, sexuality, language etc.) which is indeed about everything, like a true philosophy must be.

After defending a Master Thesis on Plato's cave under the supervision of Sarah Kofman in 1988 at the University of Paris 1 (Panthéon-Sorbonne), I was seriously thinking of doing a PhD on Schopenhauer with Clément Rosset. This did not happen because on the one hand Rosset was professor in Nice and I was in Paris (about one thousand kilometers by road) and on the other hand because I started to concentrate more and more on logic.

But the following years when doing at the same time a PhD in philosophical logic an a PhD in mathematical logic I wrote four papers on Schopenhauer: one on suicide [11], dedicated to Sarah Kofman who committed suicide on the date of Nietzsche's 150th birthday, October 14, 1994, one on Schopenhauer's criticism of the use of logic in mathematics [8], two on the principle of sufficient reason, related to a the proposal of a formalization of this principle in quantified propositional modal logic by one of my advisors: Newton da Costa ([7] and [9]). This second one is an extended abstract of a talk presented at the *38th Conference of History of Logic*, November 17-18, 1992, Kraków, Poland.

In 1992-93 I spent one year and a half in Poland (for details see [25]) and I remember that when there I read the recently published biography of Wittgenstein by Ray Monk [82] and was pleased to learn that Schopenhauer was the philosopher that Wittgenstein had read the most, extensively reading it when a teenager and re-reading it after having written *Tractatus*, before the start of his second period where he developed ideas on philosophy of mathematics influenced by Schopenhauer.

After all these years I am glad to be back to Schopenhauer and I would like to thank Jens Lemanski who invited me to take part to the event he organized at the University of Hagen, December, 7-8, 2017, *Mathematics, Logic and Language in Schopenhauer*, and to contribute to this volume. Moreover Jens made many comments on a first draft of this paper, useful for its improvement.

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