

---

# Relativizations of the Principle of Identity

DÉCIO KRAUSE, *Department of Mathematics, Federal University of Paraná, P.O.Box 19081, 81531-990 Curitiba, PR, Brazil.*  
E-mail: [dkrause@mat.ufpr.br](mailto:dkrause@mat.ufpr.br)

JEAN-YVES BÉZIAU, *Institute for Advanced Studies, University of São Paulo, Av. Prof. Luciano Gualberto, trav. J, 374, 05508-900 São Paulo, SP, Brazil.* E-mail: [beziau@usp.br](mailto:beziau@usp.br)

## Abstract

We discuss some logico-mathematical systems which deviate from classical logic and mathematics with respect to the concept of identity. In the first part of the paper we present very general formulations of the principle of identity and show how they can be ‘relativized’ to objects and to properties. Then, as an application, we study the particular cases of physics (the transgression of the principle of identity by quantum objects) and logic (some logics in which the principle of replacement is not valid are presented). In the last part of the paper, we discuss the *alphabar logics*, that is, those logical systems which violate a formulation of one of the most fundamental versions of the principle of identity; in these logics, there are formulas which are not deducible from themselves.

## 1 Introduction

“Obviousness is always the enemy of correctness. Hence, we must invent a new and difficult symbolism in which nothing is obvious.”

Bertrand Russell

One of the most ‘obvious’ principles in which classical logic is based is the principle of identity. There are several and non equivalent ways of formulating this principle, as we will recall in brief below. By the sake of correctness, perhaps it would be better to say that there are various non equivalent mathematical formulations, some of them expressed by logical means, which are usually referred to as ‘the principle of identity’.

Let us recall that every mathematical interpretation of philosophical ideas or principles is in certain sense always dubious and subject to criticisms and, despite the mathematical method has been proved to be successful in many ways, mathematical interpretations of fundamental principles of thought (or reasoning) are often more difficult and controversial. Anyway, we will try to show that such a method is fruitful also in the case of one of the most fundamental principles of logic and mathematics and even of science in general.

The principle of identity has been investigated from various points of view, such as philosophically, mathematically (such as in algebra, where the principle of identity

## 2 Relativizations of the Principle of Identity

is fundamental, as the notion of an equation shows us), as well as logically (cf. the classical books of modern logic).<sup>1</sup> In this work, we will present a rather different approach involving some new mathematical systems.

The main objective of the paper is not to investigate the principle of identity ‘directly’, but to study the possibility of obtaining some interesting and useful relativizations of this principle. We begin by giving an account on the principle of identity as inspired by Leibniz’ ideas. We present very general formulations of that principle, which are valid for objects and for properties. We also show how to relativize these formulations at the abstract level and then we present some particular applications to philosophical problems concerning both quantum physics and logic. Finally, we discuss a more fundamental and ambiguous formulation of the principle of identity: ‘every object is *identical* to itself’. In logic, an interpretation of this formulation is  $A \vdash A$  (every formula is deductible from itself); then, we present logics which violate such a principle; we call these systems *alphabar logics*.

A more detailed discussion about the relationships between the development of logics in which some ‘deviation’ from the theory of identity of classical logic and mathematics and the foundations of physics, in particular in what respects the logical basis of quantum mechanics, can be found in our papers [6], [7], [11], [18], [19], [28], [29] and in certain sense also in [3], [8], [9].

## 2 ‘Abstract’ formulation of the Principle of Identity and its relativizations

It is commonly agreed that the concept of identity formulated in classical logic and mathematics, so as in the majority of the most known logical systems, is closely related to Leibniz’ ideas [10], [4, p. 300]. This is essentially due to the manner by means of which this concept is introduced in these formalisms. Despite the differences in the definitions, due to the level of the employed language, the general idea of indistinguishability is used to define identity. In other words, in all these systems identity between objects is defined<sup>2</sup> by means of ‘agreement with respect to attributes’ or then by the fact that all that can be said of one thing can also be said of an ‘identical’ one (the substitutive law, mapped in Leibniz’ dictum “*eadem sunt quorum unum potest substitui alteri salva veritate*” [4, p. 300], [21, ft. 1], which is also a form of indistinguishability (see below).

Leibniz stated that there are no two different entities with all their attributes in common. As it is well known, he formulated his principle of the identity of indiscernibles (PII) in several places of his work, but apparently always in the negative (cf. [39]): (in short) if things are distinct (not equal) then there exists some quality which distinguishes them. The ‘translation’ of PII to the languages of logic, in particular to the ‘positive form’, is by our own (see below). Notwithstanding, even by paying the price for the lack of an adequate interpretation of Leibniz’ thought in

---

<sup>1</sup> See for instance [36, 46, 23, 39, 21].

<sup>2</sup> The word ‘definition’ must be understood here in a very general way, encompassing for instance what might be called ‘definitions by postulates’, as when we introduce (the primitive concept of) identity in first-order logics by stating the usual axioms of reflexivity and substitutivity, which in certain sense ‘define’ that concept *e.g.*, [31, p. 74]. Some authors like Savellos [37] maintain that identity cannot be defined, but must be taken as a primitive, undefinable notion. According to Savellos, all ‘definitions’ of identity are circular in the sense that all of them presuppose identity itself. Even so, the axioms state identity in a rather ‘Leibnizian’ way.

this paper, we will agree with some authors that there is a sense in saying that the concept of identity in classical logic and mathematics is ‘Leibnizian’ [9]. In reality we must be careful concerning this; the particular formulation of Leibniz’ principle in first order logic makes reference to objects which are not space-time objects, while Leibniz’ original versions of PII apparently is more concerned with ‘concrete’ objects, that is, space-time located entities.<sup>3</sup> In fact, by using the languages of modern logic, it is possible to interpret this principle as referring to objects of any kind, including ‘abstract’ objects, such as properties, numbers or other mathematical objects. From the philosophical point of view, this interpretation seems to constitute a very important generalization of Leibniz’ original principle.

### 2.1 General ‘informal’ definition of the Principle of Identity

The general ‘informal’ definition of the principle of identity (in short, PI) to be considered here is the following:

*Two objects are identical iff they share the same properties.*

This formulation is ‘neutral’ in the sense that it can be interpreted both ‘objectively’, that is, the objects’ properties do not depend on the knowledge of the objects themselves (that is, the formulation is not epistemic), and ‘subjectively’, in the sense that the identification of the objects depends on the conceptual framework of the theory. It is rather this second interpretation which leads to the notion of *indistinguishability*. In order to relativize this ‘informal’ PI, we may consider two possibilities: the relativization of PI concerning objects and the relativization of PI concerning properties.

#### 2.1.1 The ‘objectual’ relativization of PI

With respect to objects, we use the following ‘objectual’ relativization of PI:  $x = y$  iff for every property  $P$ , if  $x$  has this property, then  $y$  has it too, and there exists a property  $Q$  such that both  $x$  and  $y$  have  $Q$ .<sup>4</sup>

#### 2.1.2 The ‘conceptual’ relativization of PI

$x$  and  $y$  are identical relative to a collection of properties  $\mathcal{C}$  iff for every property  $P \in \mathcal{C}$ ,  $x$  has the property  $P$  iff  $y$  has it too.

More specific formulations of these two relativized forms of PI will be discussed in the next sections.

### 2.2 Very general ‘formal’ definition of the PI

The ‘formal’ PI to be considered here is:

---

<sup>3</sup> The discussion of this point is very interesting, but we do not intend to consider it here.

<sup>4</sup> There is a clear analogy between this formulation and the concept of *relative identity*, mainly in considering the analysis of the expression ‘ $x$  is the same  $Q$  than  $y$ ’ [20], [46, pp. 82ff].

#### 4 Relativizations of the Principle of Identity

*a is identical to b, in symbols,  $a = b$ , iff for all property  $P$ , object  $a$  has property  $P$  iff object  $b$  has it too.*

Depending on the level of the language we use, there are various different and non equivalent ways of formulating this principle, as for instance:

##### 2.2.1 Identity between formulas

For example, in the quantified propositional calculus [40]. If  $F$  and  $H$  are propositional formulas, then

$$F = H \text{ iff } \forall G(G[F] \leftrightarrow G[H])$$

where  $F$  and  $H$  are subformulas of  $G$ .

##### 2.2.2 The first-order formulation

If  $x$  and  $y$  are individual variables and  $F$  is a formula, then for all  $x$  and  $y$ :

$$x = y \text{ iff } F(x) \leftrightarrow F(y)$$

This schema encompasses an infinite collection of formulae, one for each  $F$ ; in the higher-order logics, it is possible to simplify it, as for instance in the second-order logics.

##### 2.2.3 The second-order formulation

If  $x$  and  $y$  are individual variables and  $F$  is a predicate ranging over the set of predicates, then for all  $x$  and  $y$ :

$$x = y \text{ iff } \forall F(F(x) \leftrightarrow F(y))$$

By using the concept of logical equivalence,<sup>5</sup> we may formulate the principle of identity without using specific connectives. For instance, in the Zero-Order formulation, two propositions are identical iff in any context one of them can be replaced by the other by preserving deduction. This form of the ‘substitution law’ will be discussed in Section 4.

##### 2.2.4 The set-theoretical formulation

In set-theoretical terms, for instance in ZFC, we can state the PI as follows: for every sets  $x$  and  $y$ ,

$$x = y \text{ iff } \forall z(x \in z \leftrightarrow y \in z)$$

which is a particular case of the first-order PI mentioned above. The reason for having mentioned this version explicitly is that a particular version of this principle is violated by the ‘quasi-set theories’ (see the next section).

---

<sup>5</sup> That is,  $A$  and  $B$  are logically equivalent iff from  $A$  we can deduce  $B$  and conversely, where the notion of deduction is primitive, in accordance with the ideas of Tarski.

### 3 The conceptual relativization of PI and quantum physics

The relationships between PI and physics have been pointed out in the literature from several points of view, but we will not recall all the discussion on this point here [15, 14, 27].<sup>6</sup>

Let us begin by formulating in a more formal way the conceptual relativized form of PI, very roughly stated above.<sup>7</sup>

In a convenient higher-order language, if  $x$  and  $y$  are individual variables and  $F$  is a variable ranging over a certain collection  $\mathcal{C}$  of properties of  $x$  and  $y$ , then:

$$x =_{\mathcal{C}} y \text{ iff } \forall F(F \in \mathcal{C} \rightarrow (F(x) \leftrightarrow F(y)))$$

and in this case we say that  $x$  and  $y$  are ‘identical relative to  $\mathcal{C}$ ’.<sup>8</sup>

The relativized conceptual principle of identity may be applied, at least in principle, to every domain of ‘objects’, even abstract ones as mentioned above. But let us consider the very particular case of quantum objects, commonly called ‘elementary particles’, which deviate from the ancient notion of ‘little bodies’, individuatable<sup>9</sup> entities having a permanent and well defined *genidentity*, like the physical objects described by classical physics.<sup>10</sup>

In what sense may we say that quantum objects provide a way of violating the Leibnizian concept of identity? The first answer would be the assumption that their individuality is provided by some ‘quid’, a *substratum* which transcends the collection of their properties. In this way, although two objects  $x$  and  $y$  might agree in what respects all their attributes, they could be distinguished by their quid, despite the well known philosophical controversy on the nature of such a substratum [15]. But it has been claimed that quantum objects are entities which are devoided of any kind of substratum [32]. Philosophers have also shown that the very idea of *quantum* is incompatible with ‘primitive thisness’, *haecceity* and the like [34], [35], [41].

In fact, the elementary particles are characterized in quantum theory according to their possession of different values of a certain number of constant properties, the ‘intrinsic properties’. Without any compromise with philosophical ways of thought (which will be not mentioned here), one could speculate on this topic from at least three points of view: firstly, in saying that the ‘veiled’ reality, to use a term coined by d’Espagnat (see [12]), does not show their basic elements completely; so, we could not know all the properties of the ‘particles’, and this could sustain the thesis that this is the motive we consider them in groups (of ‘identical’ particles). The second hypothesis is that the formalism of quantum theory is ‘incomplete’ in what respects individuation in not providing all the necessary tools for individuating a ‘particle’ in all situations.<sup>11</sup>

<sup>6</sup> See also the papers listed in the footnote 2.

<sup>7</sup> The formulation we use here is an adaptation of the concept of ‘relative indistinguishability’ presented in [6].

<sup>8</sup> More precisely, in the definition it should be supposed that  $\mathcal{C}$  is a predicate of type  $\langle\langle i \rangle\rangle$ , while  $F$  is a variable of type  $\langle i \rangle$  and  $i$  is the type of the individuals, where  $x$  and  $y$ . Then, instead of  $F \in \mathcal{C}$ , we could write  $\mathcal{C}(F)$ . One example of the use this principle in quantum physics is the definition of the concept of identity of elementary particles given in [22, p. 275]: in short, two elementary particles are ‘identical’ iff they agree with respect to all their state-independent (or ‘intrinsic’) properties. See [6].

<sup>9</sup> We use this word in the sense of Redhead and Teller’s works. See their works mentioned in the references.

<sup>10</sup> It is not surprising that recent literature little by little has been substituting the word ‘particle’ by others which do not suggest the intuitive idea of a *body*, despite the controversies even in this point. For instance, B. d’Espagnat, in the early 60s, have already used the word ‘quanton’ in the same context [12], that is, in trying to avoid the word ‘particle’. Teller [41] prefer to use *quanta*, despite there is no agreement concerning the terminology in the literature.

<sup>11</sup> This possibility was also supposed by van Fraassen, but he rejected it [44] –see also [45].

## 6 Relativizations of the Principle of Identity

The third possibility sustains that the very ‘nature’ of quantum objects is such that they cannot be viewed as individuable entities, even conceptually. Schrödinger, in particular, maintained this point of view [38]; see also [32] and, more recently, Redhead and Teller [34, 35]. So, from this last view, the *quanta* are characterized by no more than certain collections of attributes done by physical laws: they are *nomological*, as expressed by Toraldo di Francia [43, p. 222], [9].<sup>12</sup>

So, under the hypothesis (roughly) described above, we are committed with the following assumptions in what respects quantum objects: (1) they are objects characterized in groups, not individually, by means of certain bundles of properties. Hence, there is no *substratum* being supposed, and quantum theories might be viewed as ‘negative’ theories concerning the concept of substance [36]; (2) due to the nomological nature of quantum objects, it is not necessary to admit that quantum formalism is not incomplete with respect to individuation. In other words, the intrinsic properties characterize the entities, in the sense that such properties (physical laws) provide all information about them [42], that is, they are *all* the relevant (or ‘essential’) properties to be considered. Then, if we take the set  $\mathcal{C}$  of the above definition to mean the collection of the intrinsic properties that characterizes a certain kind of quanta, it is easy to see that the conceptual formulation of PI is incompatible with (1) and (2), for if PI in that form would be valid, then all quantum objects of the same kind would be the very same entity (since  $x = y$  means, in usual mathematics, that  $x$  and  $y$  are no distinct objects).<sup>13</sup>

### 3.1 The violation of the set-theoretical version of PI

Quasi-set theories were developed to provide a mathematical framework for dealing with collections of indistinguishable (but not identical) objects [9, 24, 25]. These theories can also be used to built structures necessary for the semantical analysis of the theories of microphysics, so as to mathematically express the quantum fact that in quantum physics there are predicates (like ‘electron’) which have no a precise reference and ‘objects’ which have no names.<sup>14</sup> One of the main motivations of these theories is that collections of quantum objects apparently do not obey the axioms of standard set-theories [25, 9, 11]. The basic motive is that collections of quantum entities seems to present an intensional behaviour rather than act as extensionally well-defined *sets*.<sup>15</sup>

An interesting result is that Leibniz’ principle of the identity of indiscernibles is violated in quasi-set theory. In fact, in this theory the following implication is not valid in general:  $x \neq y \rightarrow \exists_Q z(x \in z \wedge y \notin z)$ , where the quantifier is relativized to quasi-sets, since the set-theoretical particular instance of this rule, v.g., the expression  $x \neq y \rightarrow x \in \{x\} \wedge y \notin \{x\}$  cannot be repeated in these theories. More details can be

---

<sup>12</sup> A fourth position could be mentioned: French and Redhead [14] have shown that quantum objects may be consistently viewed as ‘individuals’ at the expenses of assuming certain restrictions on the states they can assume. But even so, as they have shown, PI is violated by bosons, fermions and also by higher-order paraparticles, treated in first-quantization. The various formulations of PII as presented by French and Redhead may be easily adapted to the formulation of the conceptual PI stated above.

<sup>13</sup> A theory where  $x = y$  can be read as meaning ‘indistinguishability’ and not strict identity was formulated in [6]. See also [25], [27].

<sup>14</sup> For details concerning these points, see [8, 9, 25, 27, 7, 11].

<sup>15</sup> It has been claimed that systems of microobjects can be described as a kind of ‘intensional-like entities’ [9] –see also [11], and even the relations between the concepts of intensions and extensions turn to be quite different from the classical cases [ibid.].

found in [11].

## 4 Objectual relativization of PI in Logic

If we consider a logic as a pair  $\langle L, \vdash \rangle$  (see [2]), we can say that two objects  $a$  and  $b$  of the domain of  $L$  are identical if and only if they are logically equivalent, i.e., if  $a$  is deducible from  $b$  (in symbols:  $\{b\} \vdash a$ ) and vice-versa. It is important to note that this schema makes sense only if the relation of logical equivalence is a congruence relation, i.e., if it is compatible with the underlying structure of  $L$ , for example, in the standard zero-order logics, the absolute free algebra  $\langle A, \langle \neg, \wedge, \vee, \rightarrow \rangle \rangle$ . In classical propositional logic, intuitionistic logic, usual modal logics (S4, etc.) and many other logics the relation of logical equivalence is a congruence relation. However, this is not the case for example in the paraconsistent logic  $\mathcal{C}_1$ . In this logic,  $a \wedge b$  for instance is equivalent to  $b \wedge a$ , but  $\neg(a \wedge b)$  is not logically equivalent to  $\neg(b \wedge a)$ . For this reason, it is not possible to algebraize  $\mathcal{C}_1$  in the usual sense, taking the quotient given by the relation of logical equivalence. C. Mortensen has shown that the only congruence relation compatible in  $\mathcal{C}_1$  is the trivial one (every object is equivalent to itself). This of course constitute a strong defect of  $\mathcal{C}_1$ , since it means that in this logic we cannot even to identify well-behaved logically equivalent formulas. Thus even since it is possible to embed classical logic into  $\mathcal{C}_1$ , we cannot deal exactly with classical logic inside paraconsistent logic as we deal with classical logic in general.

One solution to this problem has been provided by one of the authors of this paper [2, 5] with the logic  $\mathcal{C}_1^+$ , which is a nice example of the objectual relativization of the PI. In the paraconsistent logic  $\mathcal{C}_1^+$ , which is an extension of the logic  $\mathcal{C}_1$ , we can identify two well-behaved logically equivalent formulas. That is to, if we restrict the relation of logical equivalence to the set of well-behaved formulas, then it is a congruence relation.

## 5 Alphabar logics

The expression  $P \vdash P$  ( $P$  is deducible from  $P$ ) is also very often called the law of identity. This can be explained by the fact that the deducibility relation is taken to be the basic relation between propositions or between formulas. But of course to say that the relation is reflexive (i.e., the stated law asserts nothing less, nothing more than that  $P$  is deducible from  $P$ ) is not to say that it is an identity relation since not every reflexive relation is an identity relation. The confusion arises especially among philosophers and can be explained (but not justified!) as follows: sometimes the identity law is expressed by  $a = a$ , that is, every object is identical to itself, but if this is to be correct, then we must add that every object is different from any other object. That is to say, identity is the diagonal relation. Of course this definition is reasonable. But we can also say that all congruence relations are identity, the diagonal relation being the finest one.

Then, having all this in mind, we see that it is very dangerous to call ‘ $P$  is deducible from  $P$ ’ the identity law or even that it is one of the formulations of the identity law since, firstly, the deducibility relation rarely is the diagonal relation and also since it is not necessarily a congruence relation (cf. Section 4 above).

Then, we prefer to call this law the *autodeductibility law*, and in this section we will

## 8 Relativizations of the Principle of Identity

see that it is possible to built logics in which this law is not valid, named *Alphabar Logics*, in honor of the Persian logician Alfabari.

To begin with, let us consider a very natural question, which can be consider with respect to the autodeductibility law: is this law a ‘natural’ law ?

To answer this question, we firstly could find an ‘intuitive logic’ in which this law is false (this shows that it is necessary) and, secondly, it would be interesting to show that this law is not so much evident to be rejected.

Concerning the first point, consider for example a logic<sup>16</sup> which is defined in such a way so that a formula  $F$  is deductible from a theory  $\mathcal{T}$  iff there exists a consistent subtheory of  $\mathcal{T}$  such that in this theory  $F$  is classically deductible. Then, it results that  $\{F \wedge \neg F \} \vdash F \wedge \neg F$ .

On the other hand, a reason for rejecting the law of self-deductibility could be based on the desire ‘to clean formal mathematics from trivial things’. In fact, the law of autodeductibility apparently can be regarded as a triviality: if we are interested in knowing what kind of theorems can be proved in Arithmetic, we are certainly not interested in knowing that from  $1 = 0$  we can deduce  $1 = 0$ .

But this is of course not so. In what follows, we will examine the basis on which this law appears as a valid law in the usual logical systems and how it can be derogated.

### 5.1 General conceptual framework

From a very general and abstract point of view, a logic  $\mathcal{L}$  can be viewed as a pair  $\langle \mathcal{F}, \vdash \rangle$ , where  $\mathcal{F}$ , the *domain* of  $\mathcal{L}$ , is a set of objects called *formulas* and  $\vdash$  is a relation between theories (subset of formulas) and formulas. This relation is called *the deductibility relation* of  $\mathcal{L}$ . To define a logic, we may use one among various possible methods, such as the *proof-theoretical method*, the *model-theoretical method* and the *consequence operator method*, and all of them include various sub-methods.

A *logical law* is a condition on the relation of deductibility. This condition can be more or less complex; famous examples other than the law of self-deductibility are the following ones:

1. **Law of Monotony:** If  $F$  is deductible in  $\mathcal{T}$ , then  $F$  is deductible from every extension of  $\mathcal{T}$ .
2. **Cut Law:** If  $F$  is deductible from  $\mathcal{T}'$  and if each element of  $\mathcal{T}'$  is deductible from  $\mathcal{T}$ , then  $F$  is deductible from  $\mathcal{T}$ .

Given a particular method, an *external law* is a law which is induced by this method, i.e., every logic built up by this method obeys this law. On the contrary, an *internal law* is a law which is independent of the considered method. Some examples will be done below.

### 5.2 The self-deductibility law from the proof-theoretical point of view

In the proof-theoretical method, the relation of deductibility is defined through the notion of *proof*; correlated concepts are those of *axiom* and (inference) *rule*.

---

<sup>16</sup> This is a variant of the paraclassical logic presented in [33].



### 5.2.1 Frege–Hilbertian systems

An *axiom* is a formula, and a *rule* is an ordered pair whose first element is a theory, the set of *premises* of the rule, and whose second element is a formula, the *conclusion* of the rule. The notion of provability is defined as follows:<sup>17</sup>  $F$  is provable in  $\mathcal{T}$  iff there is a sequence of formulas such that  $F$  is its last member and every member of the sequence is an element of  $\mathcal{T}$  or is an axiom, or is the conclusion of a rule whose premises precede it in the sequence.

In this method, the autodeductibility law is an external law (as well as the cut law and the law of monotony). The crucial point in the definition is the concept of *to be an element of  $\mathcal{T}$* . This clause induces in fact the infinitary self-deductibility law. We can drop this clause; in this case the set of theorems (i.e., formulas which are provable from the empty set) is preserved. But a most convenient idea is to modify this definition by substituting *is an element of  $\mathcal{T}$  different of  $F$*  by *is an element of  $\mathcal{T}$* . In this case, we have in the so modified classical logic:  $F \wedge G \vdash F$ ,  $F \wedge G, F \vdash F$  and  $H \wedge G, F \not\vdash F$ . In fact, if  $F \notin \mathcal{T}$ , then  $F$  is deductible from  $\mathcal{T}$  in the orthodox way iff it is deductible from  $\mathcal{T}$  in the modified way.

### 5.2.2 Hertz–Gentzenian systems

A *sequent* is a pair of theories, in symbols:  $\mathcal{T} \rightarrow \mathcal{T}'$ . An axiom is to be understood here as a pair formed by a sequent and a rule in such a way so that the premises and the conclusion are sequents. The definition of proof runs like this: ' $F$  is provable in  $\mathcal{T}$  iff there is a sequence of sequents such that  $\mathcal{T} \rightarrow \mathcal{F}$  is its last member and every member of the sequence is an axiom or the conclusion of a rule whose premises precede it in the sequence'.

In this case, the law of autodeductibility is not external. In all these systems, this law is more or less directly induced via the definition of proof by a schema of the following type:

1. **Atomised Identity Schema:**  $A \rightarrow A$  for all atomic formulas  $A$
2. **Generalized Identity Schema:**  $F \rightarrow F$  for all formulas  $F$ .

In the majority of the systems it is not possible to reduce the generalized identity schema to the atomised one. For example, in the so-called *natural deduction systems* this reduction is not possible. But such a reduction is possible in the so-called *sequent calculus systems*, when the connective rules are symmetrical, as for instance in the intuitionistic system or in the classical system. But for example if we drop one of the negation rules of classical logic, then we obtain a system in which this reduction is not possible anymore. In the case where the reduction is possible, the law of self-deductibility is induced by the atomised axiom schema and by structural properties of the system, which preserve identity through complexification.

It is very easy to obtain alfabar logics when the reduction is not possible by rejecting the generalized schema and keeping only with the atomised one. But in this case the systems obtained are much weaker than the original ones. For example in a semi-negational system it is not possible to prove that:  $\neg F \wedge G \vdash \neg F$ .

---

<sup>17</sup> We emphasize that this definition is neither due to Frege nor to Hilbert.

## 10 Relativizations of the Principle of Identity

### 5.3 The autodeductibility law from a model-theoretical point of view

#### 5.3.1 The Bolzano–Tarskian definition

The orthodox model-theoretical definition of deductibility is the Bolzano–Tarskian one: ‘ $F$  is deductible from  $\mathcal{T}$  iff every model of  $\mathcal{T}$  is a model of  $F$ ’.

The general prerequisite for this definition is a set of models  $\mathbf{M}$  and a function  $mod$  which associates to each formula and each theory a set of models. Such a pair  $\mathcal{S} = \langle \mathbf{M}, mod \rangle$  is called *semantics*.

In the case of the Bolzano–Tarskian definition, the autodeductibility law is external, induced by the fact that  $mod(F) \subseteq mod(\mathcal{T})$ .

#### 5.3.2 Semantical crossing

A very general way of constructing alphabar logics from a model-theoretical point of view is by the method of *semantical crossing*.

Given for example two semantics  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , we have the following definition: ‘ $F$  is deductible from  $\mathcal{T}$  iff every  $\mathcal{S}_1$ -model of  $\mathcal{T}$  is an  $\mathcal{S}_2$ -model of  $F$ ’.

Then, we have an alphabar logic iff there exists  $F$  such that  $mod_1(F)$  is not included in  $mod_2(F)$ . Extending this idea of defining a model-theoretical deductibility relation by using several semantics, we can construct alphabar logics in many other ways.

#### 5.3.3 The adequation problem

In [2] we have shown that logics which verify the law of autodeductibility, the law of monotonicity and the cut law (*normal logics*) are exactly Bolzano–Tarskian logics. The question now is how to extend this result to alphabar logics in finding a general model-theoretical method to define them. We consider here *cut-logics*, i.e., logics which verify the cut law.

We construct two canonical semantics in which models are theories of the given logic and in which the modelisation functions are defined as follows:

1. In the first case,  $mod_1(\mathcal{T}) = \{\mathcal{T}' : \forall F(F \in \mathcal{T} \rightarrow \mathcal{T}' \vdash F)\} \cup \mathcal{T}$
2. In the second case,  $mod_2(\mathcal{T}) = \{\mathcal{T}' : \forall F(F \in \mathcal{T} \rightarrow \mathcal{T}' \vdash F)\}$

Then, by semantic crossing, we define the relation of deductibility:  $\mathcal{T} \vdash F$  iff  $mod_1(\mathcal{T}) \subseteq mod_2(F)$ . It is easy to see that with this definition we have a general completeness result for cut-logics, and in particular for alphabar logics which are also cut-logics.

### 5.4 The law of autodeductibility from the point of view of the consequence operator method

In the consequence operator method (due to Tarski), the deductibility relation is defined through a function  $Cn$  from the power set of the domain of the logic to itself. The usual conditions imposed on this function include the following one:  $\mathcal{T} \subseteq Cn(\mathcal{T})$ . This condition is in fact directly equivalent to the infinitary self-deductibility law. Thus, in this framework the law of self-deductibility appears as an external law. It is possible simply to throw it away and keep only with the two other laws  $\mathcal{T} \subseteq \mathcal{T}' \rightarrow$

$Cn(\mathcal{T}) \subseteq Cn(\mathcal{T}')$ . The question is to know in which extend the methods used in this framework can also be applied for this weakened function. The study of this subject has been developed by G. Malinowski [30], who has reinforced the second law in  $Cn(\mathcal{T} \cup Cn(\mathcal{T}) \subseteq Cn(\mathcal{T})$  and thus constructed the theory of a *quasi-consequence operator*.

## Acknowledgements

The authors would like to thank the anonymous referees for useful comments.

D. Krause is partially supported by CNPq (Brazil). J.-Y. Béziau is supported by the Swiss National Research Fund and by the Swiss Academy of Sciences.

## References

- [1] Bradley, F. H. (1958), *The principles of logic*, Oxford, Oxford University Press.
- [2] Béziau, J.-Y. (1995), *Recherches sur la logique universelle*, PhD Thesis, Université de Paris 7. 7, 7, 10
- [3] Bitbol, M., *Mécanique quantique: une introduction philosophique*, Paris, Flammarion, 1996. 2
- [4] Church, A. (1956), *Introduction to mathematical logic*, Vol. I, Princeton, Princeton University Press. 2, 2
- [5] da Costa, N. C. A., Béziau, J.-Y. and Bueno, O. A. S. (1995), ‘Aspects of paraconsistent logic’, *Bulletin of the IGPL* **4**:597–614. 7
- [6] da Costa, N. C. A. and Krause, D. (1994), ‘Schrödinger logics’, *Studia Logica* **53**:533–550. 2, 5, 5, 6
- [7] da Costa, N. C. A. and Krause, D., ‘An intensional Schrödinger logic’, to appear. 2, 6
- [8] Dalla Chiara, M. L. ‘An approach to intensional semantics’, *Synthese* **73**:479–496, 1987. 2, 6
- [9] Dalla Chiara, M. L. and Toraldo di Francia, G. (1993), ‘Individuals, kinds and names in physics’, in Corsi, G. et. al (eds.) *Bridging the gap: philosophy, mathematics, physics*, Kluwer, Dordrecht, pp. 261–283. 2, 3, 6, 6, 6, 6, 6
- [10] Dalla Chiara, M. L. and Toraldo di Francia, G. (1993a), ‘Identity questions from quantum theory’, preprint. 2
- [11] Dalla Chiara, M. L., Giuntini, R. and Krause, D., ‘Quasiset theories for microobjects: a comparison’, preprint. 2, 6, 6, 6, 6
- [12] d’Espagnat, B. (1965), *Conceptions de la physique contemporaine*, Paris, Hermann. 5, 5
- [13] d’Espagnat, B. (1989), *Reality and the physicist*, Cambridge, Cambridge University Press.
- [14] French, S. and Redhead, M. (1988), ‘Quantum physics and the identity of indiscernibles’, *British J. Phil. Science* **39**:233–246. 5, 6
- [15] French, S. (1989), ‘Identity and individuality in classical and quantum physics’, *Australasian J. of Philosophy* **67**(4):432–446. 5, 5
- [16] French, S. (1989a), ‘Why the principle of the identity of indiscernibles is not contingently true either’, *Synthese* **78**:141–166.
- [17] French, S. and Krause, D. (1995), ‘Vague identity and quantum non-individuality’, *Analysis* **55**(1):20–26, 1995.
- [18] French, S. and Krause, D., ‘The logic of quanta’, forthcoming in the *Proceedings of the Boston Colloquium for the Philosophy of Science*, L. T. Cao (ed.). 2
- [19] French, S., Krause, D. and Maidens, A., ‘Quantum vagueness’, preprint. 2
- [20] Geach, P. (1987), ‘Relative identity’, in Perzanowski, J. (ed.), *Essays on philosophy and logic*, Proceedings of the XXXth Conference on the History of Logic, dedicated to Roman Suszko, Cracow, Oct. 1984. Jagiellonian University Press, 1987, pp. 19–32. 3
- [21] Gardies, J.-L. (1989), ‘La définition de l’identité d’Aristote à Zermelo’, *Theoria – Segunda Época* **10**:55–79. 2, 2
- [22] Jauch, T. (1968), *Foundations of quantum mechanics*. New York, Addison-Wesley. 5

## 12 Relativizations of the Principle of Identity

- [23] Joja, A. (1962), 'Logique et métalogue du principe d'identité', *Serie Acta Logika* **5**:7–44. 2
- [24] Krause, D. (1990), *Non-reflexivity, indistinguishability and Weyl's aggregates* (in Portuguese), Thesis, Department of Philosophy of the University of São Paulo. 6
- [25] Krause, D. (1992), 'On a quasi-set theory', *Notre Dame Journal of Formal Logic* **33**(3):402–411. 6, 6, 6, 6
- [26] Krause, D., 'Axioms for collections of indistinguishable objects', preprint.
- [27] Krause, D. and French, S., 'A formal framework for quantum non-individuality', *Synthese* **102**:195–214, 1995. 5, 6, 6
- [28] Krause, D. and French, S., 'Opaque predicates and their logic', presented at the XIth Brazilian Conference on Mathematical Logic, Salvador, May 1996. To appear. 2
- [29] Krause, D. and French, S., 'Quantum objects are vague objects', preprint. 2
- [30] Malinowski, G. (1990), 'Q-consequence operation', *Reports on Mathematical Logic* **24**:49–59. 11
- [31] Mendelson, E. (1987), *Introduction to mathematical logic*, Monterey, Wadsworth & Brooks Cole. 2
- [32] Post, H. (1963), 'Individuality and Physics', *The Listener* **70**:534–537 5, 6
- [33] Puga, L. Z., N. C. A. da Costa and R. J. Vernengo, 'Sobre algunas lógicas paraclásicas y el análisis del razonamiento jurídico', to appear. 8
- [34] Redhead, M and Teller, P. (1991), 'Particles, particle labels and quanta: the toll of unacknowledged metaphysics', *Foundations of Physics* **21**:43–62. 5, 6
- [35] Redhead, M and Teller, P. (1992), 'Particle labels and the theory of indistinguishable particles in quantum mechanics', *Brit. J. Phil. Sci.* **43**:201–218. 5, 6
- [36] Quinton, A. (1973), *The nature of things*, London, Routledge & Kegan Paul. 2, 6
- [37] Savellos, E. S., 'On defining identity', *Notre Dame J. of Formal Logic* **31**(3):476–484, 1990. 2
- [38] Schrödinger, E. (1957), *Science theory and man*. London, Allen and Unwin. 6
- [39] Stroll, A. (1972), 'Identity', in Edwards, P. (ed.) *The encyclopedia of philosophy*, MacMillan Pub. Co., pp. 121–124. 2, 2
- [40] Suszko, R. (1975), 'The Fregean axiom and Polish mathematics in the 1920s', *Studia Logica* **36**:377–380. 4
- [41] Teller, P., *An interpretive introduction to quantum field theory*, Princeton, Princeton University Press, 1995. 5, 5
- [42] Toraldo di Francia, G. (1978), 'What is a physical object?', *Scientia* **113**:57–65. 6
- [43] Toraldo di Francia, G. (1981), *The investigation of the physical world*, Cambridge, Cambridge University Press. 6
- [44] van Fraassen, B. (1985), 'Statistical behaviour of indistinguishable particles: problems of interpretation', in Mittelstaed, P. and Stachow, E. W. (eds.) *Recent developments in quantum logics*. Mannheim, Bibliographisches Institut, pp. 161–187. 5
- [45] van Fraassen, B. (1991) *Quantum Mechanics: an empiricist view*. Oxford, Oxford University Press. 5
- [46] Williams, C. J. F. (1989), *What is identity?*, Oxford, Clarendon Press. 2, 3

Received 30 January 1996. Revised 20 May 1996