TRANSITIVITY AND PARADOXES

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The scene takes place in the Tea Room of the Philosophy Department of Harvard University.

Bill: Hi Tom. How's it going?

Tom: Pretty good, and you?

Bill: Not so bad, but I'm a bit perturbed by a new paradox I came across last night while reading Picsou magazine to improve my French.

Tom: I love paradoxes! I always find solutions to them. Tell me yours!

Bill: Tom, as far as I know, you're a successful philosopher and quite a rich one, too.

Tom: That's right, but I don't see any paradox there. I'm a good philosopher; I've gained recognition, and I make a good living out of it as well.

Bill: Sure, that's perfectly OK. How wealthy are you, approximately?

Tom: Let's say that my assets are around 10 millions dollars.

Bill: OK. Now imagine that you take one cent out of it and give it, say, to charity. Will you still consider yourself rich?

Tom: Sure!

Bill: Now, would you agree that a rich man having X dollars wealth, is still a rich man after we take one cent away?

Tom: I must agree with you once again.

Bill: But then, if we apply this principle a certain number of times, leaving, say, a hundred dollars left after giving a lot of cents to charity, will you still be rich?

Tom: Yes. That's a nice way to be charitable.

Bill: And even having nothing, would you still be rich?

Tom: That's right! I agree with this emphatically because I'm a Buddhist.

Bill: So you don't see any paradox?

Tom: No. You can't make a poor man out of me with your cheap argument.

Bill: But what went wrong in my argument? You agreed with it every step of the way.

Tom: Sure, every step, but not with the whole way.

Bill: What do you mean? This is probably a Buddhist metaphor, but my knowledge of Buddhism is weak. You've lost me..

Tom: So, let's take a walk together. We're now next to the office of the late Quine.

Bill: That's right, the guy who wrote an essay Entitled, "The Ways of Paradox".

Tom: Right, Quine argued that when there is a paradox, something went wrong and we have to revise our theory a bit. This means that we don't want to revise first the most fundamental principles.

Bill: Yes—that'd be too easy. The worst case scenario would be for people to work within a paraconsistent logic, positing simply that a paradox itself is not a problem. .

Tom: Lazy bastards! and then come post-modern philosophers who would ask for a revision of the Identity Principle, vive la différance!

Bill: Or those suspicious guys who say that everything is vague or fuzzy.

Tom: Absurd! I'm rich, and there's nothing fuzzy about that.

Bill: OK. So what principle should we revise, if any?

Tom: Have you heard of Alfred Tarski?

Bill: Sure. He's the guy who said that snow is white and made a whole theory of truth out of that.

Tom: Yes, a clever guy. Do you also know about his work on the consequence operator ?

Bill: The stuff Etchemendy made a fuss about and got a lot of publicity out of it?

Tom: Not exactly. In a lesser known series of papers at the end of the twenties, Tarski developed a very abstract approach to logic where he described some general features valid for any kind of deduction..

Bill: Yes, I remember. The principles are not about connectives or logical operators, but about the notion of consequence itself.

Tom: Yes. It was quite a jump into abstraction, but not completely successful, "il a trébuché", as a French philosopher would say.

Bill: What do you mean?

Tom: These abstract principles are not completely absolute.

Bill: Everything is revisable!

Tom: Yes, and one of Tarski's principles has been revised by the pinguins.

Bill: That surprises me! Quine would've been surprised too!

Tom: Here's something else to consider. Have you heard of McCarthy?

Bill: Yes, but what does Tarski's theory have to do with politics? Revisionism in politics is quite controversial!

Tom: You didn't get it, Bill. I was talking about John McCarthy, the founder of Artificial Intelligence and the developer of non-monotonic logic.

Bill: Oh, now I get the point:
Monotonicity was one of Tarski's fundamental principles.

Tom: Right. But what about the other Tarskian axioms?

Bill: Well, there are a couple of others: Reflexivity and Transitivity.

Tom: So which one did you use to make me poor?

Bill: Rejecting reflexivity would be extremely radical, because then we couldn't even deduce that you are rich from the fact that you are, indeed, certainly rich.

Tom: That'd be curious indeed!

Bill: That would be by transitivity, but I don't see how it operates here.

Tom: Then let's go back to where we were.

Bill: We didn't move. We're still next to the office of the late Quine.

Tom: Now imagine that we're taking one step away from his office. Are we still next to it?

Bill: Sure—not very far away anyway.

Tom: Now, if we're next to his office and take another small step away, we'll still be next to it, won't we?.

Bill: Quite right.

Tom: So, by taking a certain number of steps from Quine's office, we'll still be next to it, right?

Bill: Yes. And by that logic arriving at Stanford, California, we'll nevertheless be next to Quine's office!

Tom: But that, certainly, can't be the case.

Bill: So how then can this thorny paradox be solved?

Tom: Well, let's begin by recognising that the concept "next" is not a transitive relation; in other words, let's assume that if "x" is next to "y", and "y" is next to "z", "x" is not necessarily next to "z".

Bill: That sounds fairly simple, quite ingenious, in fact.

Tom: So, each step along the way is fixed:
By taking just one step away from something I'm still next to it. However—and here's the point—I can't go from Harvard to Stanford using the next relation, because it's not transitive.

Bill: And what about wealth? Are riches "non-transitive"?

Tom: The question here is not about wealth in itself, but about the fact that you used transitivity in the deduction process itself.

Bill: I didn't see that! I've been ruined by transitivity!

Tom: You take one cent out of my pocket, and you conclude that I'm still rich; then you repeat the operation, and conclude again that I'm rich; finally, after taking all the money out of my pocket, you conclude that I'm still rich.

Bill: I see now. Each step of my reasoning is correct, except for my jumping, as it were, from the initial statement that you are rich to the final assertion that you are rich even without a single pcent left in my pocket! I was using transitivity.

Tom: Yes. You could say, "Transitivity made a pauper out of me!" By the same token, transitivity makes poor people "rich" and bald people "hairy"!

Bill: That's a powerful tool!

Tom: Powerful and dangerous, leading to appalling sophisms.

Bill: But isn't transitivity the basis of most ordinary reasoning. *Modus ponens*, after all, is also a kind of transitivity, isn't it?

Tom: You're perfectly right. All syllogistic is based on transitivity:
The disappearing of the middle term is nothing but transitivity.

Bill: Barbara will not contradict you on that point!

Tom: So one could say that all our reasoning may be based on this false assumption.

Bill: That's really frightening!

Tom: Yes it is! Maybe we were wrong all along the way.

Bill: But what about Cut-Elimination?

Tom: I don't see your point

Bill: You know that Gentzen constructed a system where nothing disappears except with the use of the "cut rule". He then showed that the cut rule itself can be eliminated.

Tom: Very clever guy, this Gentzen, but if a logic generated with the cut rule is transitive, and you can show, as Gentzen did, that you can prove the same thing without the cut rule, then the new logic you've used is still transitive.

Bill: You're right; that just means that we don't need to take transitivity as a postulate, but our reasoning is nevertheless still transitive.

Tom: Now what about a system where "cut" is not eliminable?

Bill: In this case, when we take cut out of it, we can't perform the same analysis. In this instance, we have a system which is non-transitive.

Tom: So it seems that it would be no big deal to construct non-transitive logical systems—and thus solve thousands of paradoxes!

Bill: You're right, that's very interesting! How could we have been so blind as to consider transitivity a fundamental principle of logic?

Tom: But Bill, transitivity is deeply rooted in our minds through the notion of order.

Bill: Yes, relations of order are typically transitive.

Tom: But why is calling a relation which is transitive simultaneously as assumption that it's also a relation of order?

Bill: That's a caprice of mathematicians. You know how they are!

Tom: I know, but that's a precarious fantasy, because where you don't have transitivity, you're not necessarily at the same time in complete disorder.

Bill: Yes. We can be logical without being transitive.

Tom: And I can take a cent out of my pocket and still be rich.

Bill: But what if you take 30 dollars out of your pocket? Would you still consider yourself as rich?

Tom: Sure, another paradox here?

Bill: No, but if you invite me for lunch, will you still be rich?

Tom: Now I get it! All this talk was just a way to get a free lunch! You're cleverer than Fontaine's fox!

Bill: Since you're talking about Jean de la Fontaine, what about going to the French restaurant, "La tambouille"?

Tom: Nice! but that's pretty expensive.

Bill: I know, but if you take another 30 dollars out of your pocket, you'll still be rich, assuming transitivity.

Tom: OK, let's go, assuming also that you'll pay for a bottle of my favourite wine, a Chateau Bruno Latour !