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Idempotent Full Paraconsistent Negations are not Algebraizable

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Abstract Using methods of abstract logic and the theory of valuation, we prove that there is no paraconsistent negation obeying the law of double negation and such that $\neg(a \wedge \neg a)$ is a theorem which can be algebraized by a technique similar to the Tarski-Lindenbaum technique.

1 What are the features of a paraconsistent negation? Since paraconsistent logic was launched by da Costa in his seminal paper [4], one of the fundamental problems has been to determine what exactly are the theoretical or metatheoretical properties of classical negation that can have a unary operator not obeying the principle of non-contradiction, that is, a paraconsistent operator. What the result presented here shows is that some of these properties are not compatible with each other, so that in constructing a paraconsistent negation as close as possible to classical negation, we have to make a choice among classical properties compatible with the idea of paraconsistency. In particular, there is no paraconsistent negation more classical than all the others.

The incompatibility appearing here is between theoretical properties (double negation and $\neg(a \wedge \neg a)$ as a theorem) and a metatheoretical property (replacement theorem). One who chooses the theoretical properties will not be able to algebraize his system with the usual Tarski-Lindenbaum method and should use some alternative treatments such as that in da Costa [5]. On the other hand, one who chooses the metatheoretical property will have to sacrifice at least one fundamental theoretical property of negation, risking the possibility of dealing with an operator that is a modality rather than a negation.

The result presented here is of the same kind as some previous results concerning the incompatibility between the replacement theorem and the paraconsistent logic **C1** of [4]. It was soon realized that the replacement theorem is not valid in **C1**. Mortensen [14] proved that, in fact, it was impossible to define a nontrivial congruence in **C1**. Urbas [19] proved that the addition of the replacement theorem to

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C1 leads to classical logic or trivializes it. (However, Mortensen [15] and da Costa, Béziau, and Bueno [8] have presented extensions of **C1** which admit nontrivial congruences.) The question whether the replacement theorem is compatible or not with the idea of paraconsistency is one of the significant remaining problems in paraconsistent logic (Béziau [2]).

2 Basic framework Generalizing the definition of Suszko (cf. [3]), we consider an *abstract logic* (or simply, a *logic*) as a structure $\mathcal{L} = \langle \mathfrak{S}; C_n \rangle$ where $\mathfrak{S}$ is a structure of domain $\mathbb{S}$ and C_n a closure operator on $\mathbb{S}$ (i.e., for every subset A and B of $\mathbb{S}$, $A \subseteq C_n A$, $C_n C_n A \subseteq C_n A$, $A \subseteq B \Longrightarrow C_n A \subseteq C_n B$). Thus, $\mathfrak{S}$ is not necessarily an absolute free algebra as in standard propositional logics: it can, for example, be a partial infinitary algebra or a relational structure. The result that we will prove here can be applied to first- or second-order language and even to non-well-founded language as described by Lismont in [12], because it does not depend on specific features of the structure $\mathfrak{S}$, which is the mathematical expression of the *language*.

Given a family $\mathfrak{F}$ of subsets of a set $\mathbb{A}$, it is easy to see that the function φ on $\mathcal{P}(\mathbb{A}) \times \mathcal{P}(\mathbb{A})$ defined by: for every $a \in \mathbb{A}$, $A \subseteq \mathbb{A}$, $a \in \varphi A$ if and only if for every $B \in \mathfrak{F}$, $A \subseteq B \Longrightarrow a \in B$ is a closure operator on $\mathbb{A}$. Such a family will be called an *adequate bivalent semantics* ($\mathcal{ABS}$) for an abstract logic $\mathcal{L} = \langle \mathfrak{S}; C_n \rangle$ when the domain $\mathbb{S}$ of $\mathfrak{S}$ is the same as $\mathbb{A}$ and φ is the same as C_n. An element of the family is called a *bivaluation* because we consider its characteristic function.

It is easy to prove that the class of $\mathcal{ABS}$ for a given logic is not empty. This justifies the study of a logic from the point of view of its class of $\mathcal{ABS}$, called the *theory of valuation* and developed by da Costa (for an overview on the subject see [6] and [7]).

3 Preliminary results A *negation* in an abstract logic $\mathcal{L} = \langle \mathfrak{S}; C_n \rangle$ is a unary operator $\neg$ on the domain $\mathbb{S}$ of $\mathfrak{S}$ with certain properties. Until now there has been no agreement concerning these properties, but it is common, for example, to call a negation an operator not obeying the law of excluded middle, such as that involved in intuitionistic logic. Thus, it is not necessarily absurd to call a negation an operator not obeying the law of noncontradiction.

A negation $\neg$ is *paraconsistent* if and only if there exists a in $\mathbb{S}$ such that $C_n\{a, \neg a\} \neq \mathbb{S}$ (a *paraconsistent logic* is a logic with such a negation). We will state an easy result establishing a connection between this definition and the intuitive traditional formulation of the law of noncontradiction (a proposition and its negation cannot both be true) and which will also be useful for the proof of the central theorem.

Before doing so we need two more definitions. A bivaluation β in an $\mathcal{ABS}$ for the logic $\mathcal{L}$ is *singular* if and only if there exists an object a in $\mathbb{S}$ such that $\beta(a) = \beta(\neg a) = 1$. A singular bivaluation is *trivial* if and only if it is the function which gives the value 1 to any object of $\mathbb{S}$.

Proposition 3.1 *A logic is paraconsistent if and only if in all its $\mathcal{ABS}$ there is a singular nontrivial bivaluation.*

Given a logic $\mathcal{L} = \langle \mathfrak{S}; C_n \rangle$, two objects a and b of $\mathbb{S}$ are *logically equivalent* (notation: $a \cong b$) if and only if $C_n a = C_n b$. Let us say that an object a is a *theorem* if and only if $C_n a = C_n \varnothing$. Obviously we have the following proposition.

Proposition 3.2 *If a and b are theorems, then $a \cong b$.*

Moreover, given $\mathcal{S}$ an $\mathcal{ABS}$ for $\mathcal{L}$, we have this proposition.

Proposition 3.3 *$a \cong b$ if and only if for any β of $\mathcal{S}$, $\beta(a) = \beta(b)$.*

An *idempotent negation* $\neg$ of $\mathcal{L}$ is a negation such that for every object, $a \cong \neg\neg a$. An *algebraizable negation* is a negation $\neg$ verifying: for every a and b, if $a \cong b$ then $\neg a \cong \neg b$. An *algebraizable logic* is a logic such that logical equivalence is a congruence on it. It is clear that a logic in which there is a nonalgebraizable negation is not algebraizable.

In the case where the structure $\mathfrak{G}$ is an absolute free algebra or something similar, such as a first-order language, it is easy to see that a logic is algebraizable in the above sense if and only if the *theorem of replacement* (as formulated, for example, by Kleene in [11]) holds. In the general case, this notion of algebraization is equivalent to the general replacement theorem formulated by Curry and MacLane (see [9] and [13]).

Proposition 3.4 *Given an idempotent algebraizable negation, two objects are logically equivalent if and only if their negations are logically equivalent.*

4 Main result A *full paraconsistent negation* is a paraconsistent negation such that $\neg(a \wedge \neg a)$ is a theorem. Here we take $\wedge$ to denote the standard conjunction, that is, $C_n\{A, a, b\} = C_n\{A, a \wedge b\}$. It is easy to check then that for any bivaluation β of a given $\mathcal{ABS}$ for the underlying logic, we have $\beta(a) = 1$ and $\beta(b) = 1$ if and only if $\beta(a \wedge b) = 1$.

Theorem 4.1 *Idempotent full paraconsistent negations are not algebraizable.*

Proof: Let $\mathcal{L}$ be a logic with a full idempotent paraconsistent negation. Given $\mathcal{S}$ an $\mathcal{ABS}$ for $\mathcal{L}$, due to Proposition 3.1, there is a singular nontrivial bivaluation β of $\mathcal{S}$, such that $\beta(a) = \beta(\neg a) = 1$ for one object a, and we have $\beta(a \wedge \neg a) = 1$.

As $\neg$ is a full paraconsistent negation, $\neg(a \wedge \neg a)$ is a theorem, and given any object b, $\neg(b \wedge \neg b)$ is also a theorem; therefore, due to Proposition 3.2, $\neg(a \wedge \neg a) \cong \neg(b \wedge \neg b)$. Now, due to Proposition 3.4, $a \wedge \neg a \cong b \wedge \neg b$, therefore, due to Proposition 3.3, $\beta(b \wedge \neg b) = \beta(a \wedge \neg a) = 1$, thus $\beta(b) = \beta(\neg b) = 1$. This shows that β is trivial, which is absurd. $\square$

5 Applications of the theorem The paraconsistent negations of Asenjo's calculus of antinomies (cf. [1]), of D'Ottaviano and da Costa's logic **J3** (cf. [10]), and of Priest's logic **LP** (cf. [16]) are defined with Łukasiewicz's **L3** table for negation taking $\frac{1}{2}$ and 1 as designated. It is easy to see that, together with the standard conjunction (included in these logics), this defines idempotent full paraconsistent negations. Therefore these negations are not algebraizable.

Moreover, due to our result, it is easy to see that there are no extensions of Asenjo's calculus, **J3** and **LP**, in which their negations are algebraizable and still paraconsistent.

6 Generalization of the theorem This theorem depends on the definition of an abstract logic $\mathcal{L}$ as a structure $\langle \mathfrak{G}; C_n \rangle$ which is quite general; however, alternative, more general definitions can be proposed such as a structure $\langle \mathfrak{G}; X_n \rangle$ where X_n is a relation on $\mathcal{P}(\mathbb{S}) \times \mathcal{P}(\mathbb{X})$ obeying axioms extending those of C_n as studied by Scott in [18]. Our result can be extended without difficulty to such a structure.

We can also generalize this result to the case where we deal not with a closure operator but an equivalence connective, in order to satisfy relevantists who think that the closure operator is wrong (cf. Routley in [17]).

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