Non Truth-Functional Many-Valuedness

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Abstract

Many-valued logics are standardly defined by logical matrices. They are truth-functional. In this paper non truth-functional many-valued semantics are presented, in a philosophical and mathematical perspective.

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Contents

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Introduction

The aim of this paper is to present a new tool for the study of logics, the concept of non truth-functional many-valued semantics. [1]

We begin, in a first part, by a general discussion about many-valued logic [2] in order to show what is exactly the philosophical and mathematical meaning of this concept, in which position it stands in the logical space.

In a second part we recall the definition of logical matrix and the standard definition of many-valued logic based on this notion. It is important to have these definitions here in order to properly understand the distinction between truth-functional and non truth-functional many-valued semantics, and these definitions are also worthy to fix the terminology which can be misleading and ambiguous.

Notions discussed informally in the first part are given a precise meaning here and in the third part where we properly define the notion of non truth-functional many-valued semantics and give some examples of such semantics.

1 Logical matrices = many-valued logic?

1.1 Logical matrices do not reduce to many-valued logic

Many-valued logics are ones of the most famous non-classical logics. They appeared independently in the work of di erent people at the end of the XIXth / beginning of the XXth century, mainly C.S.Peirce, E.Post and J.Lukasiewicz. The work of Lukasiewicz was without doubt the most influential in the development of many-valued logics. One of the reasons is that Lukasiewicz's work promoted the concept of logical matrix, central concept for the construction of many-valued logics, implicit in the works of Peirce and Post. The concept of logical matrix was later on systematically used for the development of many-valued logic.

Moreover the notion of logical matrix has been used not only to generate many-valued logics. For example Bernays [2] used many-valued matrices at the metalogical level to prove the independence of sets of axioms for classical propositional logic. It has also been used at the metamathematical level by people like Kleene [23], Bochvar [13] or Girard [20]. It was the idea of Tarski that the concept of logical matrix could be used as a basic tool for the general theory of zero-order logics.[3] And in fact this concept really became fundamental

[1] Non truth-functional many-valued semantics were introduced for the first time some years ago in one of my papers about non classical logics [3]. But in this paper they just appear as a side notion.

In some sense the present paper is a sequel of two other papers ([6], [18]), although it is self-contained. The paper [18] arose from a discussion about G.Malinowski's book on many-valued logics [25]. This little book is a good presentation of standard many-valued logic.

[2] We will alternatively use the singular "many-valued logic" or the plural "many-valued logics", depending on what we want to emphasize. The singular expression emphasizes many-valued logics as a whole.

[3] In a footnote to the reedition of "Investigations into the sentential calculus" by Lukasiewicz and Tarski in Tarski's volume *Logic, semantics, metamathematics,* Tarski recalls that "the

in the so-called “Polish logic”.[4]

So logical matrices do not reduce to many-valued logic. But what about the converse: does many-valued logic reduce to logical matrices? To generate many-valued logics do we need logical matrices?

1.2 Logical matrices do not violate the principle of bivalence

What is the definition of many-valued logic? One can define a many-valued logic as a logic which violates the principle of bivalence. But what it the principle of bivalence? It can be stated as follows:

Principle of Bivalence *A proposition is true or false: it cannot be true and false, or neither true nor false.*

The interpretation of this principle is not necessarily obvious.[5] One can for example seriously doubt that the standard many-valued logics challenge this principle. The reason is the following: the principle of bivalence is present in many-valued matrices through the distinction between designated and undesignated values, as stressed by G.Malinowski: “The matrix method inspired by truth-tables embodies a distinct shadow of two-valuedness in the division of the matrix universe into two subsets of designated and undesignated elements.” ([25], (p.72)). This very distinction is crucial for the definitions of logical truth and logical consequence.

When we have, for example, a many-valued matrix, with three values 0, 1/2, and 1, it is therefore misleading to call the value 0 true, the value 1 false and the value 1/2, indeterminate, or true-false, etc. The designated values must be considered as corresponding to truth, and the undesignated values as corresponding to falsity, because logical truth is defined as “designated for every valuation”. The same many-valued matrix can generate totally di erent logics according to the choice of designated/undesignated values. For example Lukasiewicz took only 1 as designated, but in the constructions of paraconsistent logics people have taken 1/2 and 1 as designated (see [1], [19], [29], [30], [36], [27]). To consider 1/2 as designated but not calling it true can lead to erroneous interpretations of the logic generated by the matrix (see [12]).

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construction of many-valued systems of logic described here, are entirely due to Lukasiewicz and should not be referred to Lukasiewicz and Tarski.” ([34], p.38) But later on when the concept of matrix is introduced, he adds the following footnote: “The view of matrix formation as a general method of constructing systems is due to Tarski” ([34], p.40).

[4] About the general theory of zero-order logics, Polish logic and this terminology see [10].

[5] Very often the principle of bivalence is confused with the principle of excluded middle and sometimes with the principle of contradiction. This confusion has been discussed in [18] and [11] and will not be treated here. We have tried here to give a formulation of this principle which avoids confusions.

1.3 Suszko's distinction between algebraic values and logical values

The concept of logical consequence seems essentially bivalent (and also the concept of logical truth which is a particular case of it). If one has a consequence relation it can be interpreted semantically: T a means every model of T is a model of a, and T a means there is a model of T which is not a model of a. So to be a model or not to be a model, that is the question. No matter how "truth in a model" is defined (using several designated values, interpretations, accessibility relations, etc.), what is important is that at the end we have the dichotomy true in a model / false in a model.

In fact if we have a consequence relation, it is always possible to find a bivalent semantics for it, just by taking as models, characteristic functions of some theories. As Suszko puts it: "In short, every logic is (logically) two-valued" ([33], p.378). Suszko indeed provided a bivalent semantics for Lukasiewicz's three many-valued logic $L3$ (see [32]). This may sound paradoxical since $L3$ is called a three many-valued logic because it cannot be defined with a two-valued matrix. But Suszko's semantics is not a matricial semantics. The values 0 and 1 in this semantics are not the domain of an algebra, they are not *algebraic values* but *logical values*, following Suszko's terminology.

In the case of propositional classical logic, algebraic values and logical values coincide in some sense because the characteristic functions of the maximal theories can be viewed at the same time as homomorphisms from the set of formulas into the Boolean algebra on $\{0, 1\}$. But in most cases this does not happen. So one must make clear the distinction between logical values and algebraic values. In the case of $L3$, we have two semantics, one with *two logical* values and a semantics with *three algebraic* values. But there are logics which cannot be characterized by (finite) matrices and therefore have no semantics with algebraic values. This is the case for example of the paraconsistent logic $C1$ of da Costa (cf. [14], [4]). For this logic, a semantics with two logical values was provided [15]. Later on da Costa and his pupils developed a general theory of logic based on such kind of semantics under the name "theory of valuation" (cf. [16], [17]).

For Suszko, "any multiplication of logical values is a mad idea and, in fact, Lukasiewicz did not actualize it" ([33], p.378). But what is exactly a logical value? Does the dichotomy algebraic value / logical value admit no third possibility? The aim of this paper is to present many-valued semantics where the values are not algebraic values in Suszko's sense, they are not elements of the algebra of a logical matrix. But we will not be mad enough to call these values logical values since, because as in the matrix case, we will make a distinction between designated and undesignated values, in order to define logical truth and logical consequence.

One can wonder why introducing such kind of many-valued semantics, since every logic is two-valued. Lukasiewicz's logic $L3$ has a bivalent semantics, so why multiplying the values and introducing a three-valued semantics? The reason is that this three-valued semantics gives a totally di erent look at $L3$

and is a very useful technical tool to prove theorems of $L3$ and metatheorems about $L3$.[6] Non truth-functional (i.e. non matricial) many-valued semantics were introduced (cf. [3]) for the study of the paraconsistent logic $C1$ which doesn't have truth-functional semantics. As we have already said, this logic has a bivalent semantics. But the use of a many-valued semantics can give a better intuition of $C1$ and simplifies the proof, in the same way as the three-valued matricial semantics does for $L3$, even if this many-valued semantics is not truth-functional.

Non truth-functional many-valued semantics are a useful tool for the study of logics. As in the case of matricial semantics, the additional values are philosophically ambiguous, and in some sense they preserve the principle of bivalence through the dichotomy designated/undesignated values. But as in the case of matrix semantics it is also possible to use these non truth-functional semantics to generate logics which are many-valued in a deeper sense.[7]

2 The standard definition of many-valued logic

2.1 Logical matrices

A *logical matrix* M is a structure $A; \mathrm{D}$ where A is an abstract algebra $\mathrm{A}; fun$ and D is a subset of the domain of the algebra A.

fun is a finite sequence of finitary functions (i.e. functions of finite arity) defined on A, called *truth-functions*. The *type* of the algebra is the specification of the length of this sequence and the arity of each truth-function. Elements of A are called *values*, an element of A is called a *designated value* if it is also a member of D, *undesignated value* otherwise. Given a cardinal κ, a κ*-valued matrix* is a matrix where the domain of values is of cardinality κ.

A typical example of logical matrix is the 2-valued matrix of classical propositional logic: $\{0,1\}; \neg, \ , \ , \to ; \{1\}$. It is important to note that here, the sign "$\to$", for example, represents a truth-function and not a connective. Generally people use the same name, as we did, for truth-function and for connectives. This is a useful device but it can be sometimes misleading (see [10]). This 2-valued matrix is many-valued in the sense that "2 is many". But according to the standard convention a many-valued matrix is a matrix of cardinality superior or equal to three.

6 And vice-versa. The non truth-functional bivalent semantics for $L3$ was introduced by Suszko rather as an "exercise de style", and it didn't seem to have further utility. However this semantics was used later on to provided a sequent-calculus for $L3$ (see [9]) using the close connection between bivalent semantics and sequent calculus.

7 G.Malinowski has used many-valued matrices to define consequence relations which are di erent than the usual one, which are in some sense more many-valued (see [26]). It is possible to use non truth-functional many-valued semantics in a similar way.

2.2 Logics

There are many ways to define what is a logical structure, we consider here three basic types of logical structures: $L1 = \ F; \ \ _1$, $L2 = \ F; \ \ _2$, $L3 = \ F; \ \ _3$.

For all these structures, F is an *absolutely free algebra* of type F; *con* . An element a of the domain F is called a *formula*. *con* is a sequence of functions called *connectives* which generate F from a subset P of F. An element p of P is called an *atomic formula*. A set of formulas T is called a *theory*.

- $_1$ is a subset of F, elements of $_1$ are called *tautologies*.
- $_2$ is a binary relation between theories and formulas. It is called a *consequence relation*.
- $_3$ is a binary relation between theories and theories. It is called a *multiple-consequence relation*.

Hereafter we will use the word logic as a generic term for these three kinds of structure.

2.3 Many-valued logics generated by logical matrices

Logical matrices are used to generate logics. With a logical matrix, one can generate a logic of type 1, 2 or 3 by a uniform method.

Given a matrix $M = \ A; \mathrm{D}$, we consider the absolutely free algebra F of the same type as A and the set HOM of homomorphisms between F and A.

An element of HOM will be called a *morpho-valuation*. A function from the set P of generators of F to the domain A of the algebra A is called an *atomic morpho-valuation*. Due to the fundamental property of absolutely free algebras, any atomic morpho-valuation has a unique extension which is a morpho-valuation. Thus, it is the same to consider morpho-valuations or atomic morpho-valuations, since there is a one-to-one correspondence between them.

Using the notion of morpho-valuations, we now define sets and relations on the domain of F, which lead to the three basic types of logical structure.

For any formula a and theories T, U:

- $_1 a$ i for every morpho-valuation μ, $\mu(a)$ is a designated value.
- $T \ \ _2 a$ i for every morpho-valuation μ, if $\mu(b)$ is a designated value for every element b of T, then $\mu(a)$ is a designated value.
- $T \ \ _3 U$ i for every morpho-valuation μ, if $\mu(b)$ is a designated value for every element b of T, then there is an element c of U, such that $\mu(c)$ is a designated value.

Given a logic L, one says that a matrix M characterized L, or that M is a *characteristic matrix* for L, i L is the logic generated by M.

According to the standard definition of many-valued logic, a logic is not a κ-valued logic if it is, or can be, generated by a κ-valued matrix. If this would be the definition, classical propositional logic would be a 242-valued logic since in fact it is not di cult to see that it can be generated by matrices of any cardinality superior to one.

A logic L is said to be κ-valued i κ is the smallest cardinal such that there exists a κ-valued matrix which characterizes L. A typical example of many-valued logic is Lukasiewicz's three-valued logic, which is generated by a three-valued, and cannot be characterized by a two-valued matrix. Concerning the cardinality of "many", the same convention applies here as in the case of matrices: the two-valued classical logic is not called a many-valued logic, a many-valued logic is a logic which is at least 3-valued.

3 Non truth-functional many-valuedness

3.1 Truth-functional logics

What is a truth-functional logic? Classical propositional logic is truth-functional, but what about intuitionistic logic? The various modal logics? etc.

One could just say that a truth-functional logic is a logic that can be characterized by a logical matrix. However this definition seems too weak, since following it, quite every logic would be truth-functional: according to a famous theorem, whose original idea is due to Lindenbaum, any logic of type 1 which is *structural*, i.e. close under substitutions, can be characterized by a matrix. And this theorem can be generalized in some sense to logic of types 2 and 3 (see [37]).

A reasonable definition runs as follows: a *truth-functional logic* is a logic that can be characterized by a finite matrix. In this sense, intuitionistic, standard modal logics ($S5$, $S4$, K, etc..), the paraconsistent logic $C1$, Jaśkowski's discussive logic and many other logics are not truth-functional.[8]

3.2 Non truth-functional semantics

Following our definition of truth-functional logic, we can say that a *truth-functional semantics* is a finite matrix. According to this definition a non truth-functional semantics is any semantics which is not a finite matrix. This definition is quite vague if we don't specify what is a semantics.

We can give a very general definition of a semantics for a logic: a *semantics* is a structure $\langle R; mod \rangle$ where R is a set of objects called *representations*, and *mod* a function from the set of formulas to the power set of R, which associates to each formula a the set *mod*(a) of representations in which a is true.

[8] One can generate logics with logical matrices in di erent other ways than the one explained in the preceding section. For example, Gödel has shown that intuitionistic logic cannot be characterized by a finite matrix [21], but Jaśkowski has shown that it can be defined by a set of finite matrices [22].

For example in the case of $L3$, representations are homomorphisms from F to the algebra of the matrix and the function *mod* is defined by: μ $mod(a)$ i $\mu(a)$ is designated. In the case of modal logics, representations are frames, and the function *mod* is defined by: μ $mod(a)$ i a is true in every possible worlds of the frame μ. [9]

Let us now consider a very simple example of non truth-functional semantics, a bivalent one. We consider an algebra of formulas F built only with two connectives, $\neg$ and $\rightarrow$, and we define a set of functions B from F into $\{0, 1\}$ as follows: β B i

- $\beta(a \rightarrow b) = 0$ i $\beta(a) = 1$ and $\beta(b) = 0$
- if $\beta(a) = 1$ then $\beta(\neg a) = 0$.

The connective $\neg$ is defined by just "half" of the condition for classical negation. The logic generated by this semantics (taking 1 as designated) is called $K/2$ and has been studied in [8]. In this paper it has been shown in particular that classical logic is translatable in $K/2$. This logic is paracomplete in the sense that a formula and its negation can both be false. If we introduce a disjunction in a natural way, the formula $\neg(a \quad \neg a)$ is not a tautology.

The set of bivaluations B is not a set of homomorphisms and in particular cannot be generated from atomic bivaluations, i.e. functions from P to $\{0, 1\}$. The behavior of bivaluations for negation can be illustrated by the following table: [10]

p	$\neg p$	$\neg\neg p$	$\neg\neg\neg p$
0	0	0	0
0	0	0	1
0	0	1	0
0	1	0	0
0	1	0	1
1	0	0	0
1	0	0	1
1	0	1	0

TABLE 1

We can interpret this non truth-functional semantics saying that the value of $\neg a$ is "not determined" by the value of a: if the value of a formula a is 1, the value of $\neg a$ must be 0, but if its value is 0, the value of $\neg a$ can be 0 or 1.

[9] More about this general definition of semantics can be found in [5], [7].

[10] It would be misleading to call such a table, a "truth-table", the similarity is rather visual than conceptual, since this table does not describe a truth-function. Anyway this kind of table can be used as a decidability method. When we say that this table is an "illustration", this means precisely this: given any bivaluation of $\mathbb{B}$, its restriction to the set of formulas which appear in the first line of the table, coincide with one of the other lines of the table; and any of these lines can be extended to a bivaluation of $\mathbb{B}$. This kind of tables were presented for the first-time in [15].

However in this semantics, the behavior of the implication is deterministic in the sense that if we know the values of the two direct subformulas of a conditional, we know the value of this conditional. The behavior of bivaluations for implication can be described by the following usual table:

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

TABLE 2

3.3 Examples of non-truth functional many-valued semantics

We will explain what is a non truth-functional many-valued semantics, generalizing the preceding example of non truth-functional bivalent semantics, to a three-valued non truth-functional semantics. In a three-valued truth-functional semantics, the value of the negation of a formula is determined by the value of this formula. For example in the case of the three-valued matrix of Lukasiewicz, if the value of a formula is 1/2, the value of its negation is 1/2. Now in a three-valued non truth-functional semantics, given a value for a formula, the value of its negation is not determined.

Let us consider a three-valued non truth-functional semantics with three values $\{\%, 0, 1\}$, where only 1 is considered as designated, defined by the following conditions: τ T i

- $\tau(a \to b)$ is undesignated i $\tau(a) = 1$ and $\tau(b)$ is undesignated
- $\tau(a) = 0$ i $\tau(\neg a) = 1$.

The behavior of threevaluations for negation can be described as follows:

p	$\neg p$	$\neg\neg p$
%	%	%
%	%	0
%	0	1
0	1	%
0	1	0
1	%	%
1	%	0
1	0	1

TABLE 3

The logic defined by this three-valued non truth-functional semantics is in fact the same as $K/2$. This can be explained as follows: in the case of the bivalent non truth-functional semantics, given p and $\neg p$, there are three possibilities than can be described by the following table:

p	$\neg p$
0	0
0	1
1	0

TABLE 4

Now in the three-valued non truth-functional semantics, these three possibilities are described by the three-values, as illustrated by the following table:

p
%
0
1

TABLE 5

The reader can check that the TABLE 3 is a reduction, in this spirit, of TABLE 1. This kind of reduction can be systematized by the following definition: given a bivaluation β of the bivalent non truth-functional semantics for $K/2$ we define a threevaluation τ_β as follows:

$\tau_\beta(a)$ = % i $\beta(a) = 0$ and $\beta(\neg a) = 0$
$\tau_\beta(a) = 0$ i $\beta(a) = 0$ and $\beta(\neg a) = 1$
$\tau_\beta(a) = 1$ i $\beta(a) = 1$ and $\beta(\neg a) = 0$

It is easy to see that with this method we get a one-to-one correspondence between B and T such that: $\beta(a)$ is designated i $\tau_\beta(a)$ is designated. This proves that the logic generated by B and T are the same, namely $K/2$.

We can say that in the three-valued non truth-functional semantics for $K/2$, some information about the value of $\neg p$ is already given by the value of p. However this does not mean that the value of $\neg p$ is "determined" by the value of p. Therefore one may have some doubts about the usefulness of this semantics. The number of values has been increased and we still have indetermination, moreover the implication which was truth-functional is now getting non truth-functional, since the TABLE 2 must be replaced by the following one:

p	q	$p \to q$
%	%	1
%	0	1
%	1	1
0	%	1
0	0	1
0	1	1
1	%	%
1	%	0
1	0	%
1	0	0
1	1	1

TABLE 6

When the value of p is 1 and the value of q is undesignated, % or 0, then the value of $p \to q$ is not determined because it can be % or 0.

Let us see now a more convincing example of non truth-functional many-valued semantics. Imagine that we modify the definition of B adding the following condition:

- if $\beta(a) = 1$ and $\beta(b) = 0$ and $\beta(b) = \beta(\neg b)$ then $\beta(\neg(a \to b)) = 1$

This condition can be described by the following table:

p	q	$\neg q$	$p \to q$	$\neg(p \to q)$
0	0	0	1	0
0	0	1	1	0
0	1	0	1	0
1	0	0	0	0
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0

TABLE 7

In this table, we need to introduce not only direct subformulas but also negations of direct subformulas of the conditional.

Now we can "translate" this semantics in a three-valued non-truth functional semantics, adding to the two conditions which define T, the "translation" of the above condition:

- if $\tau(a) = 1$ and $\tau(b) = 0$ then $\tau(a \to b) = 0$

which can be described by the following table:

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$
%	%	1	0
%	0	1	0
%	1	1	0
0	%	1	0
0	0	1	0
0	1	1	0
1	%	%	%
1	%	0	1
1	0	0	1
1	1	0	1

TABLE 8

The subformula $\neg q$ does not appear neither in the above condition nor in the corresponding table. The three-valued non truth-functional semantics has the *subformula property* but not the bivalent non truth-functional semantics. That is basically what we have gained.

4 Conclusion

If this paper we have presented the concept of many-valued non-truth functional semantics, in particular comparing it to many-valued truth-functional semantics and bivalent non-truth functional semantics. We have tried to show the usefulness of this concept through an example of a three-valued non-truth functional semantics. But of course there are other examples. One can develop four-valued non-truth functional semantics, etc.

These non truth-functional many-valued semantics basically keep a bivalence feature through the distinction between designated and undesignated values, but this is also the case of the matricial truth-functional many-valued semantics, so it cannot be considered as an argument against them, unless it is also considered as an argument against standard many-valued logics.

From the point of view of truth-functional many-valuedness, a many-valued logic is a logic that cannot be characterized by a two-valued matrix. If we want to generalize this definition to non truth-functional many-valuedness, we face a problem since any logic can be characterized by a two-valued non truth-functional semantics.

Anyway it seems that the standard concept of many-valued logic is quite confuse. In fact if we do not limit the matricial definition of many-valued logic to logics that can be characterized by finite matrices, any logic is many-valued (due to Lindenbaum's theorem), except classical logic, which is not by convention, considering that 2 is not enough to be "many". It seems to us that the standard concept of many-valued logic should be withdrawn: logics which can be characterized by finite matrices should simply be called truth-functional, with the addition of the smallest cardinality for which they can be characterized by a matrix.

On the other hand the expression "many-valued semantics" should be kept but its meaning should be extended in order to include not only finite or infinite matrices, but also non truth-functional many-valued semantics.

These many-valued semantics are useful tools for the study of logics defined as sets of tautologies or consequence relations, but can also be used in a more radical way to generate logics which challenge the principle of bivalence in a deeper sense, and which truly deserve the name "many-valued logics".

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