Logical structures from a model-theoretical viewpoint

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Dedicated to Edelcio Gonçalves de Souza for his 60th birthday

Abstract

We first explain what it means to consider logics as structures. In a second part we discuss the relation between structures and axioms, explaining in particular what axiomatization from a model-theoretical perspective is. We then go on by discussing the place of logical structures among other mathematical structures and by giving an outlook on the varied universe of logical structures. After that we deal with axioms for logical structures, in a first part in an abstract setting, in a second part dealing with negation. We end by saying a few words about Edelcio.

1 Logics as structures

It is usual nowadays to consider a logic as a structure of type $\mathcal{L} = \langle \mathbb{F}; \vdash \rangle$ where
• $\mathbb{F}$ is a set of objects, called *formulas*. Sets of formulas are called *theories*.
• $\vdash$ is a binary relation between theories and formulas, i.e. $\vdash \subseteq \mathcal{P}(\mathbb{F}) \times \mathbb{F}$, called *consequence relation*.

The idea is to consider this kind of structure in the same way as other mathematical structures. A structure than can be seen as a *model* of some *axioms*, similarly for example to a structure of order $\mathcal{O} = \langle \mathbb{O}; < \rangle$ where
• $\mathbb{O}$ is a set of objects.
• $<$ is a binary relation between objects i.e., $< \subseteq \mathbb{O} \times \mathbb{O}$, called *order relation*.

The situation for logical structures is a bit ambiguous, tricky, mysterious because there is an interplay between the method used and the

objects under study. These objects are logics and the method used is part of logic, namely *model theory*. There are four mammals of mathematical logic, in order of appearance:
• Set Theory
• Proof Theory
• Recursion Theory
• Model Theory.

In modern times there has been a proliferation of *logical systems*, that can be simply called *logics* and that we consider here as *logical structures*. The study of logical systems can be called *metalogic*. It is performed using the four mammals of modern logic. Since 1993 the present author has promoted *universal logic* [3], not as one system among the jungle of logical systems, not even as a super system. Universal logic is a general theory of all these logical systems, in a way similar to *universal algebra*, which is a general theory of *algebraic systems*, or simply *algebras*. And, like in universal algebra, the idea is to consider these systems as mathematical structures. Universal logic is part of metalogic or/and a way to approach metalogic, using in particular model theory, but it can also be developed using for example category theory.

One ambiguity we are facing here is that the word *theory* is used in three different ways:
• When we are talking about model theory, the word is used in the sense of a *general scientific field*, like relativity theory, the theory of evolution or number theory.
• In model theory, a *set of axioms* that characterizes a given class of structures, is called a theory, for example a set of of axioms for lattices. This is different from *Lattice Theory*, which is the study of all the different kinds of lattices and the way they can be axiomatized.
• In universal logic we are considering structures where a *set of objects* is called a theory. This is not the case when dealing with a structure whose elements are, for example, numbers.

2 Structures and axiomatization

Model theory does not reduce to the study of logical structures, it deals with any kind of structures. There is no canonical definition of model theory. In a general perspective, we can say that model theory studies the relations between structures and axioms.

Given a class of structures, we may want to *axiomatize* it, by finding some axioms whose models are exactly the structures of this class. On the other hand, given some axioms, we can investigate the class of structures that are models of these axioms.

What is a structure? We can reply to this question in the same way as we can reply to the question *What is a cat?* by pointing at our favorite cat Miaou. Let us therefore first start with an ostensive reply, by pointing at a famous structure, the structure $\mathcal{N} = \langle \mathbb{N}, < \rangle$ where
• $\mathbb{N}$ is the set of natural numbers.
• $<$ is the relation of strict order between natural numbers.

In some sense it is quite easy to understand what it is, a 7-year old child can understand it. Natural numbers such as 0, 1, 2, 3, 4, 5, are well-known and also one can understand what a big number like 7.794.798.739 (the number of human beings on Earth, right now) is. All numbers have a name, it is not like dogs. And if we ask if $7689 < 987$ we know how to answer. We don't even need a calculator (curiously calculators generally don't make this kind of operation, maybe they think there is no operation to perform here).

A more complicated story is to find some axioms which characterize this structure. What does this mean? An order relation is transitive and anti-symmetric:
• If $a < b$ and $b < c$ then $a < c$.
• If $a < b$ then $b \not< a$.
But the relation of strict order on natural numbers does not reduce to these axioms, or, to put it the other way round, such axioms are not enough to characterize it. An additional axiom is for example the following:
• Given any number a, there is a number b such that $a < b$.
This can be expressed in a more colloquial way as:
• There is not greatest natural number.
And in a more formal way as:
• $\forall x \exists y \; xRy$.
Note that in both cases the symbol "$<$" was sent to the sky. Its presence is only in the middle way, which is generally the way of the mathematician by contrast to the butcher and the logician.

One may want to find a set of axioms that exhausts all properties of the relation $<$ of the structure $\mathcal{N} = \langle \mathbb{N}; < \rangle$. From a structuralist point of view, this also means that it characterizes the natural numbers themselves. The numbers are nothing else than their relations, a number like 9 has no inner nature, what it has, is, its position. The structuralist approach was strongly promoted by Bourbaki [11].

It is not possible to axiomatize in first-order model theory the structure $\mathcal{N} = \langle \mathbb{N}; < \rangle$. Any set of axioms expressed in first-order logic has models which are different from the structure $\mathcal{N}$. This result is due to Skolem [18]. This is an application of the compactness theorem, according to which if every finite subtheory of a theory has a model, this theory has a model.

We will not enter here in the details of such kind of result: its relation with Gödel's first incompleteness theorem and so on. But we take this example to emphasize three important characteristics of axiomatization from a model-theoretical perspective:

• Model-theoretical axiomatization is not the same as proof-theoretical axiomatization, i.e. to derive some theorems from some basic principles, called axioms.

• In the perspective of model-theory, axioms are specific cases of theories, they are finite or recursive sets of formulas.

• Axioms, as well as theories, are generally expressed in a specific formal language, the most famous one being the language of first-order logic.

Having made these clarifications, we will in the next sections present logics as structures in a model-theoretical way, studying the relation between these structures and some axioms. We will stay in the middle way of ordinary mathematics, not specifying, not formalizing too much, the language we are using for expressing the axioms. We just want to point out that if this would be formalized, it would not be naturally formalized in the language of first-order logic, because the central concept of logical structures, the notion of consequence relation, is a relation between sets and objects, typically a second-order relation, by contrast to first-order relations which are only between objects.

We conclude this section emphasizing two points. The first-point is that using logic to talk about logic can be done in a fruitful and intelligent way. Linguists use language to talk about languages, this is not a problem, there are no vicious circles if the perspective is clearly understood. For example it should be clear that general linguistics, the theory of all languages, is not itself a super language. It is expressed and developed using languages, there is no priority of a given language for doing that. The second point is that what we are doing here is not fundamentally new, it is in the line of the Polish school of logic: connected to some works of Tarski (consequence operator), Roman Suszko (abstract logic) or Helena Rasiowa and Roman Sikorski *The mathematics of metamathematics* (see [4], [6] and [9]). The difference between the approach presented here, *Universal Logic*, is that we are *thematizing* the notion of logical structure and clearly differentiating these structures from other mathematical structures, as we will explain in the next section.

3 Logical structures within the family of mathematical structures

Let us come back to our starting point:

Logical structures
A logic is a structure $\mathcal{L} = \langle \mathbb{F}; \vdash \rangle$ where
• $\mathbb{F}$ is a set of objects, called *formulas*. Sets of formulas are called *theories*.
• $\vdash$ is a binary relation between theories and formulas, i.e. $\vdash \subseteq \mathcal{P}(\mathbb{F}) \times \mathbb{F}$, called *consequence relation*.

Logical structures are part of the family of mathematical structures. Using the biological hierarchical distinction between *family*, *genus*, and *species*, we consider that logical structures are a specific genus of structures. Let us examine three other genera of the family.

Order structures
A *structure of order* is a structure $\mathcal{O} = \langle \mathbb{O}; < \rangle$ where
• $\mathbb{O}$ is a set of objects.
• $<$ is a binary relation between objects i.e. $< \subseteq \mathbb{O} \times \mathbb{O}$, called *order relation*.

Algebraic structures
An *algebraic structure* is a structure $\mathcal{A} = \langle \mathbb{A}; f_{(i \in I)} \rangle$ where
• $\mathbb{A}$ is a set of objects.
• $f_{(i \in I)}$ is a collection of functions defined on $\mathbb{A}$.

Topological structures
A *topological structure* is a structure $\mathcal{T} = \langle \mathbb{P}; \mathcal{T} \rangle$ where
• $\mathbb{P}$ is a set of objects, called points.
• $\mathcal{T}$ is a set of subsets of $\mathbb{P}$ called a *topology*.

All these four genera of structures are similar in the sense that they are made of a pair. The left part of the pairs are all the same, this is a naked set. But there is a variation on the right side, this is the essence of the genus. The right part is often called the *signature*. Two structures having different signatures are of different genera.

An order relation is not considered as of the same genus as an algebra because its signature is a binary relation, whether in the case of an algebra the signature is made of functions. The signature of an algebra may vary: it can be only one binary function or one unary function together with two binary functions, etc. So we have different species of algebras. A group is not of the same species as a ring.

It is not necessarily easy to make the difference between species and genera of structures. For example if in the definition of logical structures, we replace the signature by a binary relation on the Cartesian product of the the power set of formulas, i.e. $\Vdash \subseteq \mathcal{P}(\mathbb{F}) \times \mathcal{P}\mathbb{F}$, can we say that we still are in the same genus?

Also a structure of a particular type can be equivalent to a structure of a different type. A striking example is a result of Stone showing that a Boolean ring is the same as a distributive complemented lattice (see [19]). A Boolean structure can be presented as a structure of order or as an algebra, or as a mix.

The equivalence between structures of different types has been conceptualized in model-theory with the notion of *expansion*. Two structures are equivalent if they have a common expansion by definition up to isomorphism.

There is also a well-known correspondence between the notion of Boolean algebra and logical structures: by factoring classical propositional logic, we get a Boolean algebra. Logical structures can be "viewed" as algebras, but this is not always the case, and it is only one point of view.

4 The diversity of logical structures

There are different ways to define a logical structure. First of all let us consider three variations of the signature:

Tautological logical structures
A logic is a structure $\mathcal{L} = \langle \mathbb{F}; \mathbb{T} \rangle$ where
• $\mathbb{F}$ is a set of objects, called *formulas*. Sets of formulas are called

theories.
• $\mathbb{T}$ is a set of formulas called *tautologies.*

Consequence logical structures
A logic is a structure $\mathcal{L} = \langle \mathbb{F}; \vdash \rangle$ where
• $\mathbb{F}$ is a set of objects, called *formulas.* Sets of formulas are called *theories.*
• $\vdash$ is a binary relation between theories and formulas, i.e. $\vdash \subseteq \mathcal{P}(\mathbb{F}) \times \mathbb{F}$, called *consequence relation.*

Multiple-conclusion logical structures
A logic is a structure $\mathcal{L} = \langle \mathbb{F}; \Vdash \rangle$ where
• $\mathbb{F}$ is a set of objects, called *formulas.* Sets of formulas are called *theories.*
• $\Vdash$ is a binary relation between theories and theories: $\Vdash \subseteq \mathcal{P}(\mathbb{F}) \times \mathcal{P}(\mathbb{F})$, called *multiple-consequence relation.*

The first formulation corresponds to how logical systems were originally conceived at the beginning of the 20th century. The second approach was mainly promoted in Poland but using a different set-up (see [20]), which is the following one:

Consequence Operator
A logic is a structure $\mathcal{L} = \langle \mathbb{F}; Cn \rangle$ where
• $\mathbb{F}$ is a set of objects, called *formulas.* Sets of formulas are called *theories.*
• Cn is a binary function from theories to theories, i.e. from $\mathcal{P}(\mathbb{F})$ to $\mathcal{P}(\mathbb{F})$, called *consequence operator.*

These two set-ups are equivalent independently of any axioms. Multiple-conclusion logical structures were developed only in the 1970s (see [17]), although one may say that they already showed up in the case of Gentzen's sequent-calculus [14], but this is rather ambiguous as we will explain in the next section.

Let us point out that we have presented all these variations on the one hand without specifying the structure of the set $\mathbb{F}$, on the other hand without stating some axioms for the "thing" which appears in the signature. This is typical of the universal logic approach we have been developing, but focusing on the second type of structures, i.e. consequence logical structures. The spirit of *axiomatic emptiness* (cf. [7]) can however also be applied to other types of logical structures. Regarding the dressing of the naked set $\mathbb{F}$, there are various ways to proceed also independent of the signature. A typical dressing, for propositional logic,

is to consider the domain of the structure as follows (cf. [15]):
• $\mathcal{F}$ is an absolutely free algebra $\langle \mathbb{F}; \wedge, \vee, \neg \rangle$ whose domain $\mathbb{F}$ is generated by the functions $\wedge, \vee, \neg$ from a set of atomic formulas $\mathbb{A} \subseteq \mathbb{F}$.
We have then a mix of two kinds of structures, by putting within a logical structure an algebra. This is what Bourbaki called a *carrefour de structures*.

The model-theoretical axiomatic methodology for logical structures does not mean that we need to fix a set of axioms. It is similar to universal algebra. There are good reasons not to fix a set of axioms, both philosophically and theoretically. Let us just consider the theoretical aspect here. Among all logical structures, it is interesting to consider the two extreme cases:
• Nothing is a consequence of nothing, i.e. $\vdash = \emptyset$
• Everything is a consequence of everything, i.e. $\vdash = \mathcal{P}(\mathbb{F}) \times \mathbb{F}$
And also it is interesting to consider that the opposite of classical logic, i.e. the set-theoretical complement of the consequence relation of this logic, is a logical structure. This is what we have called *anti-classical logic* [10]. It is not possible to host all these structures, in the universe of logical structures, if we are working with a specific set of axioms.

Similarly to the universal algebra approach the universal logic approach does not mean that we will not consider axioms. But axioms are always relative and are a way to classify and study the relations between different logical structures, to navigate within the ocean of logical structures

5 Axioms for abstract logical structures

The most famous axioms for logical structures are the three following Tarski's axioms (cf. [5]):
• $a \vdash a$ (*Reflexivity*)
• If $T \vdash a$ and $T \subseteq U$ then $U \vdash a$ (*Monotonicity*)
• If $T \vdash a$ and $U, a \vdash b$ then $T, U \vdash b$ (*Transitivity*)

We call *Tarskian logic*, a logical structure obeying these axioms. These axioms were originally presented by Alfred Tarski but in a different way because, he was working with a consequence operator, not with a consequence relation. There are lots of different equivalent ways to present these axioms, even within the same type of structures. For

example sometimes reflexivity is presented as
• $T, a \vdash a$ (*Extended reflexivity*)

If we are a minimalist, it is not necessary to present it in this way because in fact this can be deduced from the three above axioms: by reflexivity we have $a \vdash a$, and since $a \subseteq \{T\} \cup a$, applying monotonicity we have $T, a \vdash a$. Since $a \vdash a$ is a particular case of $T, a \vdash a$, i.e. the case when $T = \emptyset$, the axiom $a \vdash a$ is equivalent to the axiom $T, a \vdash a$ modulo monotonicity. This means that if we replace the axiom of reflexivity by extended reflexivity we define the same class of logical structures.

We have shown that extended reflexivity is deducible from reflexivity using monotonicity. Is this a proof? Yes! But an informal proof as standard mathematicians are doing, when dealing with order structures, algebraic structures, etc. Such kind of proof cannot be easily translated into first-order logic, like in fact most of mathematical proofs, in particular here because we are using second-order structures, but this can be translated into first-order logic for example via set theory. Although we are aware that this is not straightforward and that some complications may show up, we are, to start with, not interested to work on the formalization in first-order logic of such proofs.

What is important for us here is to make a clear distinction between this structural approach and a proof-theoretical approach such as sequent calculus. There can be some confusions, which are in particular generated by terminology and symbolism. For example the above Tarski's axioms look like the so-called *structural* rules of sequent calculus. But are they the same? Can we identify the cut-rule with transitivity? This would be highly misleading.

Gentzen's sequent system LK generates a logic which is the same as the logic generated by LK^-, i.e. LK wihtout the cut-rule. The logical structure generated by LK^- is the same classical logic as the one generated by LK. The consequence relation generated by the cut-free system LK^- is transitive! This is what shows the cut-elimination theorem.

What is interesting however is that we can develop informal proofs about logical structures which are inspired or directly imported from sequent calculus and vice-versa, we can import informal proofs about logical structures within sequent calculus. This is important because this can secure an "algorithmic" aspect. Note however that even if everybody agrees about the computable aspect of first-order classical sequent-calculus, this does not mean that the metatheory of this system is itself formalized.

But the idea we are discussing here it the applications of model-theoretical methods to the study of logical structures. Let us give a fairly simple example. Consider for example the following axiom:
• If $T \vdash a$ then there is To finite, $To \subseteq T$ such that $To \vdash a$ (*Compactness*)
We can show that this axiom is not a consequence of Tarski's axioms by giving an example of a logical structure which verifies Tarski's axioms but not the axiom of compactness. An example of such a structure is second-order classical logic considered from a standard model-theoretical way.

6 Axioms for logics of pure negation

The *law* or *principle* of non-contradiction was traditionally considered as a basic principle of logic. It was considered either as a law of thought or a law of reality, or both. We will not discuss these pataphysical questions here. There are even more extravagant people considering that this law has to be rejected. We will also not comments this kind of extravagance. What is important for us here is to show how we can have a new and hopefully better understanding of this law using the model-theoretical approach, independently of wanting to approve or reject it.

Boole formulated the law of non-contradiction as $x(1-x)=0$ and showed how to deduce it from $x^2=x$, which for this reason he considered as the fundamental law of thought (see [8]). This is a purely algebraic approach in the sense that he is using functions and equalities. But we don't consider algebra as a panacea for mathematics, there are other mathematical structures, and moreover we consider that a different approach provides a better understanding.

We consider negation as a unary function $\neg$. The second step is to consider this function on a naked set, with only this function, so we have the following structure: $\mathcal{F} = \langle \mathbb{F}; \neg \rangle$. The third step, which is properly original. is to consider the structure: $\mathcal{LPN} = \langle \mathcal{F}; \vdash \rangle$. $\mathcal{LPN}$ is an acronym for *Logic of Pure Negation.* And then we consider axioms for this negation.

There are here two important features directly connected with the spirit of universal logic:
• We can work with an algebra which is not necessarily an absolutely

free algebra. We may have an algebra where $\neg\neg a$ is a, or even $\neg a$ is a.
• The axioms for the consequence relation are not absolute. Axioms for negation can be considered independently of axioms for the consequence relation.

We can consider the following axiom:

$$\text{Given } T \text{ and } a, \text{ for any } x\text{: } T, a, \neg a \vdash x$$

independently of Tarski's axioms for consequence relation. And also we can consider the relations between the above axiom and the following second one

$$\text{Given } T \text{ and } a\text{: } T, \neg a \vdash x, \text{ for any } x \text{ iff } T \vdash a$$

according to or not according to such or such axiom for the consequence relation.

We have shown that it is possible to deduce all axioms for negation from this second axiom, modulo Tarski's axioms (see [2]). Can we call this axiom, the axiom of non-contradiction? To answer this question it is important to study the relation between this axiom and the following principle:

$$a \text{ is true iff } \neg a \text{ is false,}$$

To do so we need to connect a theory of truth and falsity with general abstract logic. This has been done by Newton da Costa with his theory of valuation on which we have been working together (see [12]).

7 Dedication and acknowledgments

I am glad to dedicate this paper to Edelcio whom I have known since 1991. I met Newton da Costa in Paris in January 1991 and he invited me to come to work with him for one year at the University of São Paulo. Arriving at São Paulo's airport in August 1991 da Costa was there together with Edelcio, who was one of his students. I stayed at Edelcio's flat for a few days in the district of *Campos Elísios* (*Champs-Élysées*) and then he took me to a residence in the campus of the university.

Since then I have continuously been in touch with Edelcio, for example taking part to the jury of his PhD Student Patricia del Nero Velasco [16]. And we share some common interest: chess, Italian food

and logical structures of course. That's why I decided to choose this topic for the present paper. In particular Edelcio took part to the *1st World Congress on Universal Logic* (UNILOG'2005) that I organized in Montreux in 2005 with the help of Alexandre Costa-Leite, the editor of this volume, who was doing a PhD with me at this time at the University of Neuchâtel in Switzerland [13].

Edelcio, together with Alexandre and Hilan Bensusan, wrote a paper for the *Festschrift* volume of my 50th birthday: "Logics and their galaxies" [1]. I am glad to reward him by the present paper.

Edelcio on the way to Marmot's paradise during UNILOG'05

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