Paraconsistent logic and contradictory viewpoints

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"A complete elucidation of one and the same object may require diverse points of view which defy a unique description"
Niels Bohr

Abstract

We start by recalling the definition of contradiction from the perspective of the square of opposition, emphasizing that it comes together with two other notions of oppositions, contrariety and subcontrariety. We then introduce the notion of paraconsistent negation as a non-explosive negation; we explain the connection with subcontrariety and why it is better not to talk of contradiction in case of paraconsistent negation.

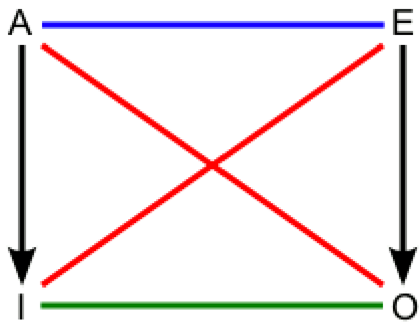
We then explain that we can interpret the paradoxical duality wave/particle either as a subcontariety in reality or as different contradictory viewpoints. We go on developing a logic based on a relational semantics with bivaluations conceived are viewpoints and in which we can define a paraconsistent negation articulating the oppositions between viewpoints.

After proving some basic results about this logic, we show the connection with modalities: we are in fact dealing with a reconstruction of S5 from a paraconsistent perspective and our paraconsistent negation is the classical negation of necessity. We finish by presenting a hexagon of opposition describing the relations between this negation, the negated proposition, classical negation and necessity.

1. Contradiction and the square of opposition

According to a general informal definition, two propositions *p* and *q* are said to be *contradictory* iff they cannot be true together and they cannot be false together. In this case we say that *p* contradicts *q* and that *q* contradicts *p*, and also that *p* and *q* together is a *contradiction*.

Within the theory of the square of opposition, contradiction is an opposition among other ones: contrariety and subcontrariety. Two propositions are said to be *contrary* iff they can be false together but not true together. Two propositions are said to be *subcontrary* iff they can be true together but not false together. Contradiction is considered as the strongest opposition and the relations between the three oppositions can be nicely expressed by a diagram :



In this diagram we have represented contradiction in red, contrariety in blue and subcontrariety in green. The black arrows are implications, traditionally called subalternations, and the four letters A, E, I, O, which also are traditional notations, can be considered as four propositions (More about the square of opposition, its history, philosophy and technical aspects, can be found in the two recent books: Beziau and Payette, 2012; Beziau and Jacquette 2012) .

What kind of example of contradiction can we give? Is the pair «Kelly is sad» and «Kelly is happy» a contradiction? One may argue that these two propositions are just contrary because Kelly may be neither sad nor happy. One may also argue that these two propositions are subcontrary, because Kelly may be sad and happy at the same time, smiling and crying as it sometimes happens. One may also even claim that Kelly can be neither happy nor sad and both happy and sad. We are then in a paranormal situation: «Kelly is sad» and

«Kelly is happy» are neither contrary, nor subcontrary, nor contradictory. Such kind of situation is not described in the square and one may wonder if in this case there really is an opposition between the two propositions (about paranormal negation and paralogics see Beziau 2012b).

Let us have a look at less controversial cases where no emotions are involved, let us enter the realm of mathematics. We can consider the two following propositions: «*K* is a circle» and «*K* is a square». Using the theory of the square of opposition, we see that these two propositions are in fact just contrary because *K* can be neither a circle, nor a square, a triangle for example. Extracting properties from these two propositions, we can say that a round square is not a contradictory object, but just a contrary object.

Let us try to find a better example: «*K* is odd» and «*K* is even». An integer can in fact not be both odd and even. And it has to be odd or even: there is no third possibility, you can divide it by two or not, even an eccentric integer like zero does not escape to it. Now if we consider that *K* is a number which is not an integer, the situation is not clear, maybe we are facing just contrary propositions, like when we are going out of the realm of mathematics: if *K* is Kelly, she is neither odd nor even (unless we are think of dividing her in two).

To find "real" contradictory propositions we maybe have to go at a superior level of abstraction, to avoid ambiguity due to vagueness and/or contextualization of concepts. We can consider a proposition and its negation, like «Kelly is happy» and «Kelly is not happy». Is it possible to argue that these two propositions are not contradictory. In fact everything is possible, up to the limits of absurdity and triviality.

This is the way to paraconsistent logic.

2. Contradiction and paraconsistent negation

The idea of parconsistent logic is to reject the principle according to which from a proposition and its negation we can deduce any proposition, a principle traditionally called *ex-falso sequitur quodlibet* and in modern times, *explosion principle*. This principle can be symbolically represented as follows:

$$p, \neg p \models q$$

The symbol $\neg$ represents negation and the symbol $\models$ represents semantical consequence. Following the standard Tarskian notion of consequence (a general notion not limited to a special logical system)

$$p, \neg p \not\models q$$

means that there is a model (world, valuation) in which p and $\neg$ p are both true and q is false.

What is called a *paraconsistent negation* is a negation not obeying the principle of explosion. If p and $\neg p$ are both true, then we have no contradiction according to the standard definition of contradiction as it appears in particular in the square of opposition. If we stick to this definition, in the case of a paraconsistent negation we can have a situation where «Kelly is happy» and «Kelly is not happy» are both true without any contradiction. But why then some people say that paraconsistent logic is the logic of contradiction?

One can decide to say that p and $\neg p$ is a contradiction because $\neg p$ is the negation of p. This is an alternative definition of contradiction. The first definition does not involve negation but is based on truth and falsity. This second definition does not involve truth and falsity, but is based on negation. In case of classical logic the two definitions coincide, but if we change the properties of negation, they are not anymore equivalent. To avoid any ambiguity it is then better to use two different terminologies, *truth contradiction* for the first definition which does not depend on negation, and *negational contradiction* for the second one.

We may wonder if in the case of a negational contradiction which is not a truth contradiction, we really are dealing with a negation. We are touching here the question of what a negation is. One can define negation using truth contradiction, saying that $\neg$ is a negation if and only if p and $\neg p$ is a truth contradiction (and there is a general result showing that in this case the negation is necessarily classical – see Beziau 2006b). Then a paraconsistent negation is not a negation and there is no paraconsistent logic as argued by Hartley Slater (1995).

If we want to support paraconsistent logic we have to find a good argument to defend the idea that a negational contradiction is still a negation and this is directly related to the question if it makes sense in general to speak about contradiction in case of negational contradiction. If we consider a modality like possibility, symbolically expressed by $\Diamond$, the two propositions p and $\Diamond$ p (like «Kelly is happy» and «It is possible that Kelly is happy») can both be true together but it makes no sense to say that p and $\Diamond$ p is a contradiction and that $\Diamond$ is a negation. $\Diamond$ is a just a unary connective. Many people define paraconsistent negation only by the negative principle $p, \neg p \not\models q$. But this is a quite naive way to proceed thinking that ¬ is a negation only because we are using the symbol of negation. Possibility also obeys this negative principle: $p, \Diamond p \not\models q$. As emphasized in the papers "What is paraconsistent logic ?" (Beziau 2000) and "Are paraconsistent negations negations?" (Beziau 2002a) one need also to have some positive principles ensuring us that ¬ is a negation.

It has been pointed out (Beziau 2003) that we can use the theory of the square of opposition for that. The fact that there may be different types of negations can be based on the square of opposition itself, which is presenting different kinds of oppositions. We can consider that to the three types of oppositions correspond three types of negations: *contradictory negation*, *contrary negation* and *subcontrary negation*. Since the word "contradiction" is in this perspective attached to only one notion of opposition, it is better to say that p and $\neg p$ forms a contrariety when ¬ is a contrary negation and forms a subcontrariety when ¬ is a subcontrary negation, reserving the name contradiction for the case where ¬ is a contradictory negation.

Within this framework a subcontrary negation is a paraconsistent negation, because according to the rejection of the explosion principle, a proposition and its paraconsistent negation can both be true and this is exactly the property that has a subcontrary negation due to the very idea of subcontrariery. This does not mean that:

subcontrary negations = paraconsitent negations.

It is possible to consider paraconistent negations which are not subcontrary negations, the most famous being de Morgan negation, a negation according to which a proposition and its negation can both be false and can both be true (about de Morgan negation, see e.g. Beziau 2009).

3. Viewpoints and the reality of opposition

In modern physics we have a subcontrary opposition between wave and particle in the sense that the proposition «*K* is a particle» and «*K* is a wave» can both be true but cannot both be false. But why should we consider this as an opposition? If we are saying that «Kelly is a girl» and «Kelly is a psychologist» are both true, it seems that there is no opposition. Is this because these two propositions can be both false, for example if Kelly is a cat?

It seems that there is something more: one may say that with particle and wave there is an opposition because something cannot be at the same time a particle and a wave, due to the very nature of wave and particle, in the same sense that something cannot be a square and a circle. But why then can we say that «*K* is a particle» and «*K* is a wave» can both be true but not that «*K* is a circle» and «*K* is a square» can both be true ? In fact it is also possible to say that these two geometrical propositions are both true, but from a different perspective, which is not the usual flat one.

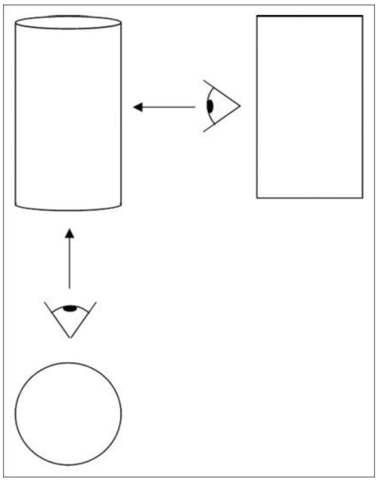
To explain the paradoxes of modern physics, the Danish physicist Niels Bohr has developed the idea of *complementarity*. He argues that there are no direct contradiction: from a certain point of view *K* is a particle, from another point of view *K* is a wave, but these two contradictory properties appear in different circumstances, different experiments. Someone may ask: what is the absolute reality of *K,* is *K* a particle or is *K* a wave? One maybe has to give away the notion of objective reality. Here is how Heisenberg, a friend of Bohr who is also considered as defending the so-called Copenhagen interpretation of quantum physics, expresses the situation: "As a final consequence, the natural laws formulated mathematically in quantum theory no longer deal with the elementary particles themselves but with our knowledge of them. Nor is it any longer possible to ask whether or not these particles exist in space and time objectively."

This quotation can be found in the famous Sokal's hoax (Sokal 1996) criticizing post-modern philosophy which according to him develops an absurd relativism abusively based on modern science. Someone may want to defend objective reality saying that reality is contradictory, reality is made of "quantons" which are contradictory objects having both the characteristics of being wave and particle. Let us note first that following our previous analysis these are not contradictory objects, but in the best cases subcontrary objects: a wave is not a particle, but "not" here is a subcontrary negation, a paraconsistent negation. If one considers that "not" is here a classical negation then he is trivializing reality: quantons are not only particle and wave, they are also flowers, stones, hamburgers and imaginary numbers.

But it is also possible to defend the idea of objective reality not considering that the reality itself is weakly contradictory, i.e. subcontrary, but that "real" opposition only appears at the level of the description of reality.

This in fact the idea defended by Niels Bohr himself : “A complete elucidation of one and the same object may require diverse points of view which defy a unique description” (cited by Sokal 1996).

It is possible to explain this situation with a simple diagram:



The same object, a cylinder, can be seen as a square and as a rectangle. There is an opposition between these two viewpoints. The opposition between round and oblong is part of the table of opposites attributes to the school of Pythagoras and is of the same kind as the opposition between a circle and a square (reducing the length of the cylinder we would have in fact a square on the right). But there is no contradiction in the cylinder itself. It is a reality of a higher dimension: living in the three dimensional space by contrast to the circle and the rectangle which are living in bi-dimensional flatland.

David Bohm, another famous physicist of the XXth century, has used a similar metaphor to explain another paradoxical phenomenon of modern physics: inseparability, i.e., the fact that two separate objects in different location can have immediate interaction. Bohm uses the metaphor of an aquarium in a room with a fish filmed by two perpendicular cameras. In another room someone may see the two movies on two different flat screens trying to understand the strange immediate interaction between the two fishes (see Bohm 1980 and Beziau 1986).

In both case what is used is the metaphor of a reality that we do not directly perceive, contrasting with the visions we have of it which are at a different level, the distinction between levels being metaphorically expressed by the difference between dimensions of space. One may think that later on we will be able to have the right concept to describe the quantons, going beyond the apparent contradiction wave-particle, or that our thought is too limited to understand reality and that we are obliged to stay forever with contradictions reflecting our incapacity to properly capture reality. Anyway we have here an interesting perspective to understand contradiction without considering that contradiction is at the heart of reality. This approach is quite general and encompasses the case of information: data about the same reality can be contradictory, this does not mean that reality itself is contradictory.

Following this idea we can develop a logical theory where the central concept is the concept of *viewpoint*.

4. Viewpoints and possible worlds semantics

From a certain viewpoint V1, *K* manifests the feature of a particle, but it does not manifest the feature of a wave. So from the point of view of V1, «*K* is a particle» is true and «*K* is a wave» is false, no contradiction here. And if we consider that physically speaking a wave cannot be a particle, we have that: «*K* is a particle» is true and «*K* is not particle» is false. The "not" here therefore behaves classically: V1(p)=1 and V1($\neg p$)=0. From another viewpoint V2, K manifests the features of a wave, but it does not manifest the feature of a particle, we can say that in this case that «*K* is a particle» is false and «K is a wave» is true, which leads to «*K* is not a particle» is true. The "not" here also behaves classically, but we have the converse situation: V2(p)=0 and V2($\neg p$)=1.

From the perspective of classical propositional logic this is not necessarily strange, viewpoints can be considered as bivaluations, and for any atomic proposition like «K is a particle», there is at least one bivaluation according to which it is true and at least one bivaluation according to which it is false. Atomic propositions are in this sense *contingent*. This is the theory of logic atomism presented by Wittgenstein in the *Tractatus* (1921), where he calls bivaluations, *truth-possibilities*. Later one they were called by Carnap (1947), *possible worlds*, and this was the second step in the direction of the so-called *possible worlds semantics*.

As we have stressed in another paper (Beziau 2006c), the main characteristic of possible worlds semantics as developed by Kripke is not the introduction of possible worlds themselves - they are already there in the semantics of classical propositional logic as bivaluations, but the relation between them - the *accessibility* relation. Such a relation can be used to define connectives in a different way than the standard one, by double recurrence, the truth-value of a proposition in a given bivaluation being defined sing the truth-value of this proposition in another bivaluation. This is the way the modalities necessity □ and possibility ◊ are defined: □*p* is true in a given bivaluation (or possible world) iff it is true in all bivaluations related to it and ◊*p* is true in a given bivaluation iff it is true in at least one bivaluation related to it. This is the main idea of such kind of semantics which are also called *relational* semantics, emphasizing their central feature. In the theory of relational semantics one studies the interaction between the properties of the relation of accessibility and the connectives generated using it. The simplest case is a universal relation where the possible worlds are all related, which is equivalent as considering no relation at all and gives rise to the modal logic S5.

We will use the strategy of relational semantics but will here consider bivaluations as viewpoints, this is less cosmic than possible worlds, however more relevant for what we want to do. We are not multiplying the worlds but just saying that the same world can be seen or conceived from different viewpoints and we are using these different viewpoints to reason about this one world.

5. Defining a paraconsistent negation using viewpoints

We will introduce a negation ~ defined in the following way: in a given viewpoint V1

$$\boxed{\text{V1}(\sim p)=0 \text{ iff } \text{V}(p)=1 \text{ for any viewpoint V related to V1}}$$

This means that the negation of *p* is false in V1 iff *p* is true from all viewpoints related to V1. So if there is a viewpoint related to V2 according to which *p* is false, then the negation of *p* is true in the viewpoint V1, i.e. V1(*p*)=V1(~*p*)=1. Let us apply this strategy to quantum physics: from a certain viewpoint (experiment) V1 «*K* is a particle» is true, but from another viewpoint (experiment) V2 related to V1 «*K* is a particle» is false , then according to V1 «*K* is not a particle» is also true. We can discuss the nature of relations between viewpoints but here to simplify we will consider that we have a universal relation, which is the same has having no relation. So we can reformulate our definition as follows:

$$\boxed{V1(\sim p)=0 \text{ iff } V(p)=1 \text{ for any viewpoint } V}$$

«*K* is a particle» and «*K* is not a particle» is not a contradiction in a given viewpoint V1: they can both be true. To show that these two propositions really forms a subcontrariety we have to prove that they cannot be false together in a given viewpoint. Imagine there is a viewpoint V7 according to which V7(p)=0 and V7(~p)=0. If V7(~p)=0, by definition this means that in any viewpoint V, V(p)=1, in particular we have v7(p)=1, which is absurd.

The negation we have introduced ~ is a paraconsistent negation. Let us now see what are the properties of such a negation, if it really deserves the name negation. The idea is to study this negation on the basis of positive classical logic, i.e. we have the connectives of disjunction, conjunction, implication, and bi-implication defined by classical conditions. This logic has been introduced in (Beziau 2006a) under the name Z and has been studied furthermore by other authors in particular using different kinds of accessibility relations (see Mruczek-Nasieniewska and Nasieniewski, 2008, 2009; Omori and Wagarai, 2008).

What we have seen therefore is that in the logic Z the negation ~ obeys the law of excluded middle in the sense that from any viewpoint V($p\vee\sim p$)=0. It is also possible to check that the following formulas are tautologies:

$$\sim\sim p\rightarrow p$$

$$\sim(p\wedge q) \longleftrightarrow \sim p\vee\sim q$$

$$\sim(p\vee q)\rightarrow\sim p\wedge\sim q$$

We let the reader check the validity of the first and third formulas. Let us consider the second one. If V7(~($p\wedge q$))=1, this means that there is a viewpoint, V5 such that V5($p\wedge q$)=0, therefore that V5(p)=0 or V5(q)=0. Consider that V5(p)=0 (Ii V5(q)=0, we have a similar reasoning) then V(~p)=1 for any viewpoint, in particular V7 and, then V7($\sim p\vee\sim q$)=1, consequently V7 (~($p\wedge q$)$\rightarrow\sim p\vee\sim q$)=1.

Let us see the other direction: If V7($\sim p\vee\sim q$)=1 then V7(~p)=1 or V7(~p)=1. Consider the first case; the other will lead to the same result. This means that there is a viewpoint, say V5, according to which V5(p)=0. We have then V5($p\wedge q$)=0. Since there is a viewpoint according to which $p\wedge q$ is false, we can conclude that V7(~($p\wedge q$))=1.

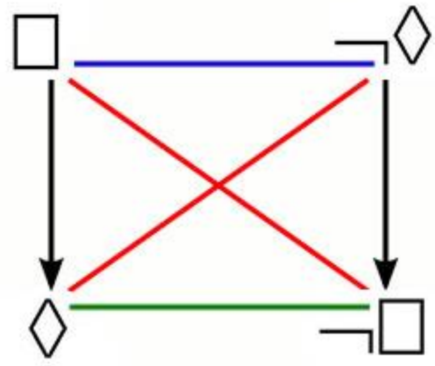
Now let us show that ~$p\wedge$~$q\rightarrow$ ~($p\vee q$) is not a tautology. We will show this through an example. Imagine that from a certain viewpoint V7 it is false that it is not the case that *K* is a rectangle or a circle. We can then infer that from all viewpoints (it is true that) *K* is a rectangle or a circle. But this does not mean that *K* is a rectangle from all viewpoints and this also does not mean that *K* is a circle from all viewpoints. It can be the case that from a viewpoint V1 it is true that *K* is a rectangle but false that it is a square and that from a viewpoint V2 it is false that *K* is a rectangle but true that it is a circle. So from the viewpoint V7 it can be true that it is not the case that K is a rectangle and it can be true that it is not the case that V7 is a circle.

Let us see now that ~p and ~~p cannot be true together. If we have a valuation, say V7, such that V7(~~p)=1, then it means that there is a valuation, say V5, such that V(5) (~p)=0, and then for any V, V(p)=1, but then V7(~p)=0. We can then therefore a classical negation putting $\neg p$=$p\rightarrow$(~$p\wedge$~~p).

What we have seen is that the negation ~ despite being paraconsistent has many properties of classical negation. So we are in a situation quite different from the operator of possibility ◊ which does not obey the explosion principle but has nearly no positive properties corresponding to negation. However the negation ~ has some relations with this modality, this is what we will now see.

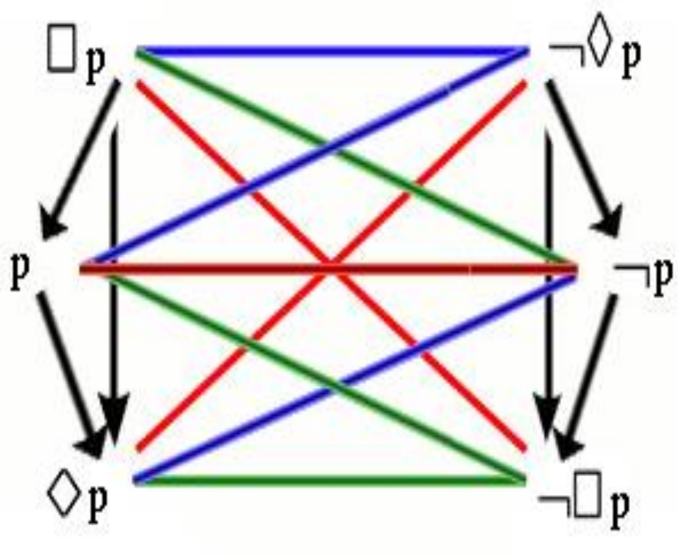
6. Paraconsistent negation and modalities

It is easy to check that our paraconsistent negation ~ behaves like the compound modality ◊¬ in S5 (¬ being classical negation): V1(◊¬p)=0 iff for all viewpoints V, V(¬p)=0 iff for all viewpoints, V(p)=1. And we know that ◊¬ is equivalent to ¬□, the modality which is at the O-corner of the modal square of opposition:

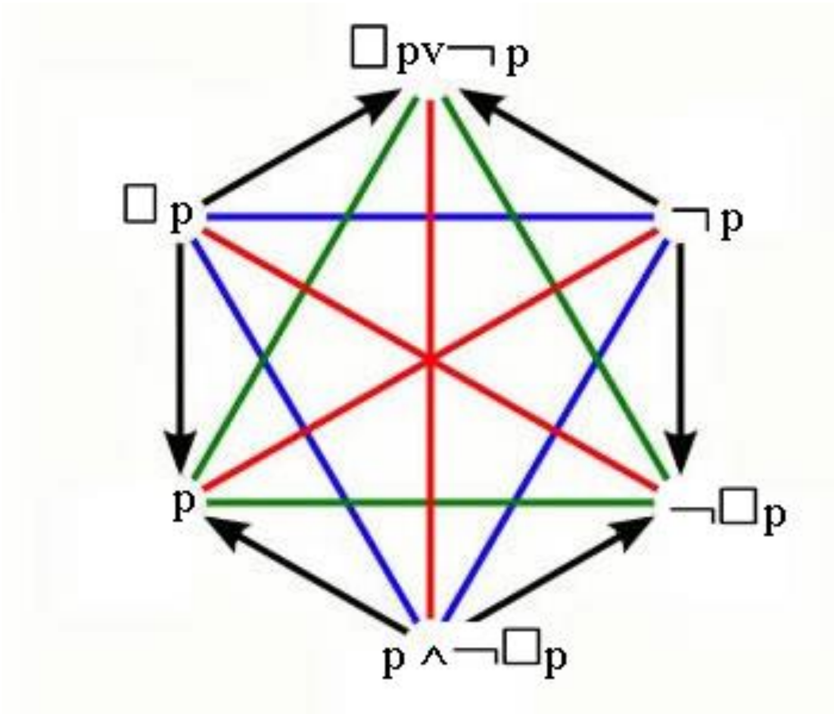


The logic Z can be directly considered as a fragment of S5: S5 with conjunction, disjunction, implication and the modality ¬□. But since we have seen that it is possible to define classical negation in Z, then it is not difficult to define the other modalities**:** for example □*p* will be defined as ¬~*p*. At the end this means that Z=S5. So Z is not strictly speaking a new logic, but a new way to define S5, putting forward the fact that ¬□ can be seen as a paraconsistent negation.

If we look at the above square of modalities, we see that ¬□*p* is subcontrary to ◊*p*. But for us what is important with ~ (i.e. ¬□ **)** is that ~*p* is subcontrary to *p*. This can be represented by the following diagram where we have added both *p* and ¬*p*:



Following the idea of the hexagon of opposition (see Blanché 1966, Beziau 2012a), which is a generalization of the square, we can place this paraconsistent negation in the following hexagon:



In this hexagon appears at the bottom the conjunction of *p* and its paraconsistent negation ~*p* (aka ¬□**)** which is not a contradictory opposition but a subcontrary opposition, as we have emphasized at the beginning of our paper, and which can be used to symbolize the paradoxical duality wave/particle understood as different viewpoints about the same reality.

7. **Bibliography**

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