What is paraconsistent logic ?

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Abstract

One can wonder whether a negation not obeying the principle of non contradiction, a paraconsistent negation, still deserve the name "negation". After briefly stating a general framework to investigate the problem, we examine what "not obeying the principle of contradiction" means and the correlative definitions of paraconsistency. We then insist on the fact that a paraconsistent negation cannot be defined only negatively, that it must obey some positive properties which characterize it as a negation and that it is difficult to specify these positive properties, in particular because different interesting properties are not necessarily compatible. To illustrate this phenomenon, we give some examples of different paraconsistent logics having some incompatible properties.

Maybe no set of postive properties is sufficient to ensure that a paraconsistent negation will be a negation ; but, anyway, paraconsistent negations are not connectives reducible to classical connectives, so that it is erroneous to say that paraconsistent logic is just a confusion between contradictories and subcontraries, as it has been recently claimed by Slater. Moreover paraconsistent logic is, in any case, a part of the theory of negation and more generally it provides an enlightenment of some fundamental features of logic ; it may also have a wide range of interesting applications.

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1 Reasons to ask the question

Paraconsistent logic can be considered as a bunch of logical systems in which there is a connective which does not obey the principle of non contradiction; such a connective is usually called a paraconsistent negation and one main problem is to know if it is legitimate to call such an operator a negation.

In a recent paper, Slater argues that in some way it is not, and that "there is no paraconsistent logic" ([37] p.451). Slater's argument is based on the same concepts as a former argument by Priest and Routley according to which they claim that one of the most famous paraconsistent negations (da Costa's C1 negation) is not a negation ([32], p.165).

So, even though paraconsistent logic is an important technical field of research officially recognized and classified under the label $03B53$ by the *Mathematical Reviews*, it is still a philosophically controversial subject : on the one hand some people think that paraconsistent logic is a mere illusion based on terminological confusion, on the other hand there are several gangs of paraconsistentists, in contradiction with each other, denying the label to their opponents.

These philosophical discussions are often quite confused because precise definitions of what are a paraconsistent negation and a negation are not stated. So it seems crucial to investigate these definitions.

In this paper we try to see how the problem of whether a paraconsistent negation is a negation can be tackle, rather than to solve it, and we will try to stay at a neutral philosophical level, only suggesting various possible philosophical options which can be taken in view of the mathematical data.

2 How to ask the question

For our discussion we need a framework which is precise but also very general. We want to study paraconsistency with a minimum of framework commitments.

<u>What is a logic ?</u>

By a *logic* $\mathcal{L}$ we mean a structure of type $\langle \mathbb{L}; \vdash \rangle$.

$\mathbb{L}$ is any set, objects of this set are called *formulas* and denoted by a, b, c,...; sets of formulas are called *theories* and denoted by T, U, V, ...

$\vdash$ is a relation between theories and formulas and is called *relation of deducibility.*

$T \vdash a$ means that $\langle a; T \rangle \in \ \vdash$ and $T \nvdash a$ means that $\langle a; T \rangle \notin \ \vdash$. If $\emptyset \vdash a$, we simply write $\vdash a$, and say that a is a *theorem* (of $\mathcal{L}$). An expression of the type $T, a \vdash b$ is used as an abbreviation for $T \cup \{a\} \vdash b$.

At the most general level, no axioms for the relation of deducibility are specified. But we recall the three famous following axioms that will sometimes be used:

For every formulas a and b, theories T and U,

(Reflexivity) If $a \in T$ then $T \vdash a$

(Monotonicity) If $T \vdash a$ and $T \subseteq U$, then $U \vdash a$

(Transitivity) If $T \vdash a$ and $U, a \vdash b$, then $T, U \vdash b$

A logic obeying these three axioms is called a *normal* logic.

Usually the domain of a sentential logic is constructed in a special way: it is an absolutely free algebra generated with functions called *connectives* from a set of formulas called *atomic formulas*.

We will not here indulge ourself with this framework commitment, but just consider that connectives are functions.

What is a negation ?

For us a negation will be, at least, a unary function [1] on the domain of a logic, denoted by the symbol $\neg$. Given a formula a, the formula $\neg a$ is called the *negation* of a.

Some people (cf. [28]) have constructed (paraconsistent) logics in which $\neg a$ can be identical to a. The above definition does not exclude this possibility, in opposition to the standard approach where, given an atomic formula a, we have: $a \neq \neg a \neq \neg\neg a \neq \neg\neg\neg a$...

What is a classical negation ?

We consider the following definition of classical negation:

$$T, \neg a \vdash x \text{ ,for all x,} \qquad \text{iff} \qquad T \vdash a$$

This definition is *pure* in the sense that it involves not other connectives. There are many other equivalent pure definitions (cf. [6]).

Semantics

We consider the notion of semantical consequence as defined by Tarski (cf. [40]): a formula a is a semantical consequence of a theory T iff all models of T are also models of a.

This definition is independent of the nature of the models and we will not need to know what is a model, but just to know what "to be a model of" means.

We define here a *semantics* for a logic $\mathcal{L}$ as a set W of objects and a function mod from the set of formulas into the power set of W. Given an element w of W and a formula a, we say that w is a model of a iff $w \in mod(a)$. A model of a theory is a model of each of its components. If w is a model of a, we say that a is true in w and conversely we say that a is false in w if w is not a model of a.

Using Tarski's definition, with a semantics for a logic $\mathcal{L}$, we get a relation between theories and formulas. If this relation is the same as the relation of deducibility of $\mathcal{L}$, we say that this semantics is an adequate semantics for $\mathcal{L}$.

The relation of consequence defined with a semantics in which there is the trivial model (w is model of all formulas) is the same as the one defined with this semantics minus the trivial model and vice-versa: if we add the trivial model to a semantics it will not modify the induced consequence relation. Therefore we

[1] One can argue, with good philosophical reasons (cf. [7]), that there are no reasons to consider that a negation should be a function and not a relation of any other kind. However, to simplify the present discussion, we will not examine the case where a negation is not a function.

can suppose without loss of generality that the trivial model is never part of a semantics.

It is easy to see that any semantical consequence relation obeys the three above axioms of normality. So a logic must be normal in order to have an adequate semantics (in this sense). However it has been shown that normality is also a sufficient condition for a logic to have an adequate semantics. (see e.g. [19]).

Due to this result, when dealing with a normal logic, we can always speak in semantical terms. For example, if in a logic $\mathcal{L}$, there are formulas a, $\neg a$, b such that:

$$a, \neg a \nvdash b$$

we can interpret this as follows:

a and $\neg a$ can both be true and b false

This is taken as an abbreviation for : in any given adequate semantics for $\mathcal{L}$, there is a model of a and $\neg a$ in which b is false.

3 Negative Criteria

3.1 Three basic definitions

Despite the many divergences among paraconsistentists there is a common agreement to found paraconsistent logic on the rejection of the following principle:

Informal ex-contradictio sequitur quod libet

From a contradiction it is possible to derive everything.

There are various ways of formalizing this principle, a standard one is the following:

Formalized *ex-contradictio sequitur quod libet* (EC)

For any theory T, formulas a and b: $T, a, \neg a \vdash b$

The same principle, without the mention of T, is clearly a particular case of it. These two principles are equivalent under monotonicity. In order to minimize framework commitments, it is therefore better to take the above formulation.

Definition of paraconsistency based on the rejection of EC

In a given logic, a "negation" is "paraconsistent" iff there exists a theory T and formulas a and b such that: $T, a, \neg a \nvdash b$.

A "logic" is "paraconsistent" iff it contains a paraconsistent negation.

There is a defintion of paraconsistency which is quite the same and has been formulated, independently, by Jaśkowski [23] and da Costa [16].

Jaśkowski-da Costa's definition of paraconsistency

In a given logic with a negation, a theory T is

- *inconsistent* iff there is a such that: $T \vdash a$ and $T \vdash \neg a$
- *trivial* iff for every a: $T \vdash a$

- *paraconsistent* iff it is inconsistent and non trivial.

A *paraconsistent logic* is a logic which can be used for the study of paraconsistent theories. [2]

If EC is rejected in a given logic, it is easy to see that this logic admits a paraconsistent theory. But if we have a logic with a paraconsistent theory, we need transitivity to ensure that EC is rejected. Thus, in some sense, Jaskowski-da Costa's definition is less general. However one can say that it has a stronger philosophical appeal.

In one of his first papers [15] da Costa claimed "every theory is permissible, since it is not trivial". This means that non triviality is the basic notion and in particular is more fundamental than consistency, to which it is not always equivalent. Cantor's naive set theory, i.e. the abstraction axiom without restriction, is inconsistent if we use classical logic as the underlying logic, but maybe there is an interesting logic in which this theory is not trivial.

Semantical definition of paraconsistency

In the case of a normal logic, the rejection of EC can be equivalently formulated in semantical terms as follows:

There is a model and a formula a such that a and $\neg a$ are both true in this model.

If we consider that a formula and its negation cannot be both true in any model, then from the standard semantical consequence viewpoint EC holds, due to the fact that the empty set is included in any other set. [3] Conversely, if EC holds in a logic, in any adequate semantics for this logic, a formula and its negation cannot both be true in a given model. [4]

The principle of contradiction

The above reasoning shows that, in case of normal logics, EC is equivalent to a staightforward interpretation of the following traditional formulation of the principle of non contradiction (PC):

A sentence and its negation cannot both be true.

This equivalence justifies the commonly informal way of speaking of a paraconsistent logic, as a logic derogating the principle of non contradiction.

However, one has to be careful, because at the time of Whitehead and Russell's *Principia Mathematica*, $\vdash \neg(a \wedge \neg a)$ would have been semantically interpreted exactly as the above formulation of PC. [5] But we now know that $a, \neg a \vdash b$

[2] We recall that neither Jaśkwoski nor da Costa originally used the word "paraconsistent" which was introduced by Quesada in 1976. Da Costa was already using "trivial" and "inconsistent" but Jaśkowski was using instead "over-complete" and "contradictory".

[3] Sometimes EC is called *ex-falso sequitur quod libet*, we can see here a possible justification for this terminology.

[4] In particular, because we have excluded the trivial model from the definition of semantics.

[5] The abusive way of speaking in the *Principia Matematica* related to these kinds of interpretations was already strongly criticized by Leśniewski. He said that it was maybe good for "épater les bourgeois" but that it was not correct (cf. [27]). Defining a paraconsistent logic as a logic derogating the principle of non contradiction seems also very good for "épater les bourgeois", but this is not necessarily incorrect.

and $\vdash \neg(a \wedge \neg a)$ are not equivalent : in the three-valued logic of Łukasiewicz [29] , EC holds but $\neg(a \wedge \neg a)$ is not a theorem ; in Priest's paraconsistent logic LP (Priest 1979), EC does not hold but $\neg(a \wedge \neg a)$ is a theorem.

In view of Łukasiewicz's logic, the rejection of the theorem $\neg(a \wedge \neg a)$ is not sufficient to get paraconsistency (founded on the rejection of EC). On the other hand one can argue that there are no reasons to reject $\neg(a \wedge \neg a)$ as a theorem to found paraconsistency and that this rejection is based on the above Russellian confusion. However there might be some other reasons to reject $\neg(a \wedge \neg a)$ as a theorem. Anyway, when defining informally a paraconsistent logic, as a logic derogating the principle of non contradiction, the point must clearly be made that it is not equivalent to say that $\neg(a \wedge \neg a)$ is not a theorem of the logic. [6]

To conclude : the three basic definitions of paraconsistency, the one based on the rejection of EC, the one based on the distinction between triviality and inconsistency and the one based on the rejection of the semantical formulation of the principle of contradiction are equivalent in case of normal logics.

3.2 Refinement of the defintion

In Johansson's minimal logic [24], EC is not valid but $T, a, \neg a \vdash \neg b$ is valid. This is clearly a particular case of EC, and we will denote it by EC#. Most people think that paraconsistency must not only exclude EC but also EC#. But in this case why not rejecting other specifications of EC, such as $T, a, \neg a \vdash \neg\neg b$ or $T, a, \neg a \vdash b \rightarrow c$? Following this idea, I.Urbas [42] has given a formulation of EC describing all these particular cases and proposed a new definition of paraconsistency based on the rejection of specifications of EC.

<u>Urbas's formulation of EC</u>

For any function f definable with connectives, for any theory T and formulas a, $\neg a$, a_1, ..., a_n:

$$T, a, \neg a \vdash f(a_1, ..., a_n)$$

So for each function f, there is a specific formulation of EC, f-EC. If f is such that for any formulas a_1, ..., a_n, $f(a_1, ..., a_n)$ is a theorem, we say that f-EC is a tautological specification of EC.

<u>Urbas's definition of strict paraconsistency</u>

A negation is strictly paraconsistent if no non tautological specifications of EC hold for it.

Da Costa's logic C1 [16] and Priest's logic LP are strictly paraconsistent but M.Urchs [44] has noticed that the standard interpretations of Jaskowski's logic are not strictly paraconsistent. So, someone who would like to keep Jaśkowski's logic within the realm of paraconsistency, should not choose Urbas's definition.

[6] So the situation here is different from the case of the principle of excluded middle : with a few suppositions it is equivalent to say that there exists a sentence which is false as well as its negation and that $a \vee \neg a$ is not a theorem.

But he probably would not think that Johansson's negation is paraconsistent, so he would have to state which are the specifications of EC he wants to reject.

It seems that the careful study of the various possibilities between strict paraconsistency and the rejections of EC and EC# is a challenging open problem.

Further Rejections ?

It would be possible to argue that a paraconsistent negation should not obey the following principle, which looks also like a particular case of EC:

$$T, \neg a, \neg\neg a \vdash b$$

However there are paraconsistent logics such as Sette's logic [36] and some formalizations of Vasiliev's logic (cf. [34]) in which this principle holds.

In fact it holds in all paraconsistent logics which are based on the idea that only atomic formulas have a paraconsistent behaviour. If one thinks that this idea is valuable, there are therefore no reasons to reject this principle.

3.3 Insufficiency of negative criteria

All the criteria presented until now are *negative* criteria in the sense that they specify principles that must be rejected.

This is clearly insufficient, because according to these negative criteria a lot of unary connectives are paraconsistent, for example all the standard modal operators are paraconsistent. The identity function is also paraconsistent in this sense, so classical logic is paraconsistent.

The problem can therefore be put as follows: on the one hand we must specify negative criteria in order to get a *paraconsistent* negation, on the other hand we must specify positive criteria in order to get a paraconsistent *negation*.

Speaking of a paraconsistent negation just as a unary connective not obeying EC (or other principles) without stating any positive principles is a highly ambiguous and controversial way of speaking, [7] [8] just fine for "ébahir le péquenaud".

4 Positive criteria

4.1 Maximality

One positive criterium can be the following:

da Costa's classical positive criterium

Add all classical properties which are compatible with the rejection of EC.

[7] In particular, if positive properties for a paraconsistent negation are not stated, the historical discussion about what was the first paraconsistent logic is meaningless.

[8] Due to the variables sharing condition, a relevant negation will not obey EC and other specifications of EC, but this does not mean that all relevant logics are paraconsistent logics.

[9] [10] (cf. e.g. [18]).

Da Costa constructed his paraconsistent logic C1 trying to catch as many classical properties as possible. But in fact the negation of C1 does not meet the above requirement as noticed by several people (e.g. [38]). There are various ways of strengthening the negation of C1. But here we see the main difficulty : these various options are not necessarily compatible.

We can thus replace da Costa's criterium by the following one:

Principle of maximality

A paraconsistent negation is a connective not obeying EC which is maximal in the sense that any strengthening of it leads to classical negation or trivialize it (i.e. trivialize the logic).

However this notion of maximality is not so clear because properties of negation can be of very different kinds. There are properties which are logical (such as e.g. $\neg\neg a \vdash a$, others which are metalogical (such as self-extensionality [11] or the possibility of defining a classical negation). There are properties which are pure (in the sense that they deal only with negation) and properties ruling the behaviour of negation with regards to other connectives (such as De Morgan's laws). It is therefore not necessarily obvious how to classified properties of a negation according to a notion of order expressing the notion of strength. To illustrate this difficulty we will examine some examples.

4.2 Compatibility and incompatibility

Choosing between connectives

Paraconsistent classical logics are paraconsistent logics which have exactly the same theorems as full propositional classical logic (cf. [39]). Examples of these logics are the dual intuitionistic logic LDJ studied by Urbas [43], the polar logic of Bychovski [14] and da Costa's paraclassical logic. [12]

In these logics it is not possible for the implication to obey the following law:

$$\text{if } T \vdash a \to \text{b then } T, a \vdash b$$

because, as we have $\vdash a \to (\neg a \to b)$, using two times this law we will get EC (without framework commitments).

[9] In order to simplify, we state the negative criterium as the rejection of EC ; this does not mean that we think that it is not necessary to reject also EC#, other specifications of EC or other principles ; here we just follow da Costa's original idea. However the validity of EC# is compatible with the rejection of EC ; in what follows, sometimes we will speak rather informally about the rejection of EC, but this shall mean, at least the rejection of EC and EC#.

[10] Jaśkowski apparently had also in mind a similar principle, but he didn't formulated it so explicitly (see [23]).

[11] To say that a negation is self-extensional is to say that if two formulas are logically equivalent, their negations too. To say that a logic is self-extensional is to say that the replacement holds for it. This terminology is used in particular in Poland (cf. [45]).

[12] In paraclassical logic, a formula is deducible from a theory T iff, in classical logic, it is deducible from a consistent subtheory of T or is a member of T.

We *must* therefore choose between paraconsistent logics in which there are such implications (e.g. classical and intuitionistic implications) and paraconsistent classical logics.

Choosing between metalogical and logical properties

Urbas (cf. [41]) has proved that if self-extensionality is added to C1, we get classical logic ; but Sylvan (cf. [38]) has shown that this is not the case of C_ω (the least logic of da Costa's C-hierarchy of paraconsistent logics), and studied the logic CC_ω, which is a self-extensional extension of C_ω. C1 and CC_ω are two incompatible extensions of C_ω, one logical extension and one metalogical extension. [13]

In [9] it has been shown that in a normal logic, a negation cannot be simultaneously idempotent (i.e. involutive), full ($\neg(a \wedge \neg a)$ is a theorem) and self-extensional. So if we want a paraconsistent self-extensional negation, we have to reject a standard logical property of negation or an axiom of normality.

Many people think that self-extensionality is an essential feature of a logic, and that therefore a logic such as C1 is not a good paraconsistent *logic* (cf. [38]).[14]

In [8] it is argued that the common defense of self-extensionality is based on a reductionist point of view (reduction of logic to algebra) and therefore is highly subject to criticisms.

In [20] it has been shown that it is possible to construct a normal logic, called overclassical logic, with a classical implication, in which classical logic is translatable, and with a paraconsistent negation which is idempotent, complete, and obeys all De Morgan's laws. This paraconsistent negation, is not self-extensional but has strong logical and metalogical properties.

Semantical properties

One could also argue that C1 is not a good paraconsistent *logic* because it has no truth-functional semantics (i.e. it cannot be characterized by a finite matrix). But why should a paraconsistent logic, or a paraconsistent negation, be truth-functional ? A lot of logics which are recognized as such, like intuitionistic logic and most modal logics, are not truth-functional.

One can argue that Priest's logic LP has a truth-functional semantics (which is in fact the same as Kleene's three valued-logic [26], the difference being in the choice of the designated values) but on the other hand has no "good" implication.

It seems that before saying that such property is good or bad for fuzzy philosophical reasons, there are serious technical investigations that have to be carry out in order to evaluate the difficulty. For example : which properties of a paraconsistent logic are compatible/incompatible with truth-functionality or with self-extensionality ?

Logical properties of negation incompatible with paraconsistency

[13] Of course, the distintion between logical and metalogical properties is relative, because for example we get C1 from C_ω by adding more logical properties, but it results also more metalogical properties, in particular the fact that classical logic is translatable into C1.

[14] Note that it has also been shown that Priest's LP is not self-extensional (cf. [35] or [9]).

One interesting question is to know which of the pure logical principles for negations are compatible with the rejection of EC (or specifications of EC).

Curry [22] has studied systematically four kinds of negations (intuitionistic negation, Johansson's negation, strict negation, classical negation). Each of these negations obeys at least one of the following versions of the laws of *Reductio ad Absurdum* and contraposition:

(RA) if $\neg a \vdash b$ and $\neg a \vdash \neg b$, then $\vdash a$
(RA#) if $a \vdash b$ and $a \vdash \neg b$, then $\vdash \neg a$

(CP) if $\neg a \vdash \neg b$ then $b \vdash a$
(CP#) if $a \vdash b$ then $\neg b \vdash \neg a$

(CP') if $a \vdash \neg b$ then $b \vdash \neg a$
(CP'#) if $\neg a \vdash b$ then $\neg b \vdash a$

In [5] we have presented a reformulation of Curry's theory of negation, studying negation independently of other connectives, and examining systematically the relations between some traditional laws for negation.

It results from this study that all the above laws are incompatible with paraconsistency in the following sense:

From RA, with finite monotonicity, we get EC, under the same condition, from RA# we get EC#, from CP we get EC, from CP' we get EC, from CP# we get EC# and from CP'# we get EC#.

These negative results can be seriously used to argue that a unary connective not obeying EC and EC# cannot be a negation, and that therefore there are no paraconsistent negations. For example, the arguments in favour of the expression "minimal negation" for qualifying Johansson's negation would militate against "paraconsistent negation".

4.3 Translatability

A property that can be considered important for a paraconsistent logic is the possibility of translating classical logic in it. C1 has this property. But this condition of translatability does not guarantee at all the strength of a (paraconsistent) negation. In [4] it has been shown that it is possible to translate classical logic in a sublogic of C1 where the paraconsistent negation is very weak. So translatability can perhaps be considered as a necessary condition but not at all as a sufficient condition for the strength of a paraconsistent negation (on this subject see also [10]).

It is not possible to fully tanslate classical logic into LP, because in this case there will be an implication in LP verifying : $\vdash a \to (\neg a \to b)$ and: if $T \vdash a \to b$ then $T, a \vdash b$; therefore, as we have seen in section 4.2, LP will not be paraconsistent. The same can be said about all paraconsistent classical logics.

C1 and LP are both sublogics of classical logic, but as classical logic is translatable into C1, C1 is in some sense stronger than classical logic and also stronger than LP.

4.4 Hard paraconsistency

One can say that a paraconsistent negation should be a unary connective not obeying EC (and EC#, etc.) but obeying a bunch of positive properties in order that it deserves the name negation.

Generally, one imagines that these positive properties are also properties of classical negation, so that a paraconsistent negation is an intermediate connective between a connective not obeying EC and a classical negation.

But this excludes the case of a negation not obeying EC and obeying properties which are not valid for classical negation. Let us call this kind of negations, *hard paraconsistent negations.*

A typical example is the following: [15]

Strong paraconsistency

A strong paraconsistent negation is a negation such that there exists a formula a such that: $\vdash a$ and $\vdash \neg a$.

Freudian negation

Another example will be a negation obeying the following principle based on Freud's notion of denegation: $T, \neg a \vdash a$.

The problem is still here to know which properties of negation are compatible with such hard paraconsistent negations.

For example, Curry's law: if $T, \neg a \vdash a$ then $T \vdash a$ is clearly incompatible with Freudian negation.

In the context of hard paraconsistency, we can keep as a fundamental idea for paraconsistent negation the principle of maximality of section 4.1 (the way that we have formulated this principle does not imply that a paraconsistent negation is necessarily weaker than classical negation).

One can even define negation with this principle, saying that a negation is a unary maximal connective. This definition would include classical negation as a particular case (cf. Post's theorem). The question will be then to know which kinds of relative maximality one will admit, i.e. one will think that they preserve the idea of negation.

4.5 Monism and pluralism

There are various incompatible positive properties that can be chosen for a connective not obeying EC.

One can choose a set of properties in order to define paraconsistent negation and argue that this is the right choice, so that the other incompatible possibilities do not define paraconsistent negations. If these arguments are not convincing, this monist choice will appear rather as dictatorial.

One can also have a more democratic attitude saying that various incompatible paraconsistent negations can all be citizens of the Realm of Paraconsistency.

[15] Mortensen [30] has shown that C1 has no (non trivial) hard paraconsistent extensions. The referee notifies that connexive logics are hard paraconsistent logics.

Allowing contradictory negations to live together fits well with the spirit of paraconsistency.

In this second case, what will support the definition of a paraconsistent *negation* is a notion of level of strenght, taking in account that various incomparable levels of strength can be chosen.

In both cases, there is still the problem of knowing whether the positive criteria really allow one to speak about a paraconsistent *negation*. In the next section we will deal with this problem by examining Slater's argument.

5 Is a paraconsistent negation a contradictory forming relation?

In [37] Slater presents an argument according to which he claims that a paraconsistent negation cannot be properly called a negation. This argument is related to a former argument by Priest and Routley (cf. [32]) according to which they claim that the paraconsistent negation of C1 is not properly a negation. But Slater uses also this argument to claim that the paraconsistent negation of LP is not a negation.

Both arguments are based on the traditional notions of contradictories and subcontraries. Let us quote Slater:

> If we called what is now 'Red', 'Blue', and vice-versa, would that show that pillar boxes are blue, and the sea is red? Surely the facts wouldn't change, only the mode of expressions of them. Likewise, if we called 'subcontraries', 'contradictories', would that show that 'It's not red' and 'It's not blue' were contradictories? Surely the same point holds.

In order to evaluate such an argument we must have a definition of contradictories and subcontraries. Here are the traditional definitions (cf. e.g. [25], p.56):

- a and b are *contradictories* iff they cannot both be true and cannot both be false.

- a and b are *subcontraries* iff they cannot both be false.

But what do true and false mean here ? It depends on the semantics. In a bivalent semantics, this is clear. In a multi-valued semantics, one can say (i) that a formula is true under a given valuation when its value is designated and false when its value is not designated, (ii) one can also adopt an alternative solution saying, for example, given three values, 0, $\frac{1}{2}$, 1 (1 and $\frac{1}{2}$ designated), that under a given valuation, a is false if its value is 0, true if its value is 1, and undetermined (or true-false) if its value is $\frac{1}{2}$. But of course one has to *choose* between the two options (i) and (ii).

It is by confusing these two options that one can say that the negation of LP is a *contradictory* forming relation and a *paraconsistent* negation, using the following reasoning:

- a formula a and its negation cannot be both true or both false in LP according to the definitions of true and false of (ii) (true is then 1 and false 0); we recall that the paraconsistent negation of LP is defined by the following conditions: $v(a) = v(\neg a) = \frac{1}{2}$ or $v(a) \neq v(\neg a)$.

- EC does not hold in LP according to the definitions of true and false of (i) (true is then 1 and $\frac{1}{2}$, false is 0).

However if one chooses (i), the negation of LP is paraconsistent, but not a contradictory forming relation and if one chooses (ii), it is a contradictory forming relation but not paracaconsistent.

In view of this analysis, Slater is right when saying that LP's negation is not a contradictory forming relation. But it has been shown that in a normal logic the only contradictory forming relation is classical negation (cf. [11]). [16] Thus in this case, to say that a paraconsistent negation is not a negation because it is not a contradictory forming relation is just to say that a paraconsistent negation is not a negation because it is not a classical negation, and this applies to any deviant negation.

But is a paraconsistent negation, such as the negation of C1, simply a subcontrary forming relation ? C1's negation is a subcontrary forming relation, but it is not only this, it has some additional properties which are not reducible to the notion of subcontraries, at least the classical notion of subcontraries. Due to the fact that the paraconsistent negation of C1 is not definable within classical logic, such a negation cannot be reduced to a given subcontrary forming relation of classical logic. [17]

Thus paraconsistent logic is not a result of a verbal confusion similar to the one according to which we will exchange "point" for "line" in geometry, but rather the shift of meaning of "negation" in paraconsistent logic is comparable to the shift of meaning of "line" in non-euclidean geometry. [18]

The question whether the shift of meaning of "negation" in paraconsistent logic permits us to still use the same word "negation" is another problem. And even if one admits that the word "negation" should be restricted to classical negation, this does not mean that paraconsistent logic is worthless.

6 Why paraconsistency is worthy

Philosophical discussions about the principle of non contradiction and negation cannot nowadays be based on traditional logic, ignoring the technical results of

[16] In this paper this result and the ideas of the present section 5 are exposed in details.

[17] However we have recently shown that it is possible to define a paraconsistent negation within the modal logic S5 or within first-order classical logic (cf. [12]).

[18] It is important to recall here that the Russian logician N.A.Vasiliev, who is considered as a forunner of paraconsistent logic, was from the same city as Lobatchevski and suggested to call a logic derogating the principle of non contradiction, an imaginary logic, in the same way that Lobatchevski called his geometry, imaginary geometry. The original terminology of Lobatchevski is quite forgotten nowadays and the terminology "non-euclidean" prevails, but here also there is a similarity because Vasiliev also called his logic, non-aristotelean logic (on Vasiliev see e.g. [2]).

modern logic and in particular the results of paraconsistent logic.

A theory of classical negation can be done, using the analytic method, by decomposing its various properties and studying the relations among them and other logical or metalogical properties. Such kind of analysis provides important results that any philosopher of logic should know, for example that most of the well-known paradoxes, like Russell's paradox, the liar paradox, etc., are not contradictions, in the sense that they can be derived without the use of a negation (by the Curry-Moh Shaw Kwey's method). From this point of view, paraconsistent logic strongly contributes to the theory of classical negation, as this paper clearly shows.

Paraconsistent logic is not only an important contribution to the theory of negation but also to logic in general. Modern logic has contributed to elucidate a lot of traditional notions, in particular by making distinctions between logical and metalogical properties, such as the difference between implication and inference. The distinction between triviality and inconsistency is also a distinction between these two levels, and although it was already known before paraconsistent logic, this latter has strongly contributed to its understanding.

An analysis of the concept of maximal consistent set, a concept fundamental for the proof of the completeness theorem, shows that the notion of negation is not at all essential to it, so that a very abstract form of the completeness theorem does not depend on classical negation or on any properties of negation, and can therefore be applied to paraconsistent logic. Generally, tools which have been used and developed for the study of paraconsistent systems, such as non truth-functional bivalent semantics (cf. [19]) or as Curry's algebras (cf. [17]), are tools which have general abstract features and that have an importance for the general study of logic.

During the nineteenth century a lot of "abnormal" algebraic structures were developed, and this led to new algebraic tools and methods and finally to a general abstract theory of algebras, called *Universal Algebra*. In a similar way, the proliferation of non-classical logics during the twentieth century leads naturally to a general abstract theory of logics, that we have proposed to call by analogy, *Universal Logic* (cf. [6]). Algebraic operations with strange behaviours have led to a very liberal and abstract conception of algebra. In the same way, we can say that strange connectives like paraconsistent negations (whether or not this terminology is appropriated) lead to a very general conception of logic.

So even if someone finds convincing arguments and technical results showing that a paraconsistent negation cannot have enough properties to really be called a negation, this will not dismiss the important contributions of paraconsistent logic to the theory of negation and to logic in general.

For sure it is absurd to call a paraconsistent *negation* any connective not obeying EC, but paraconsistent logic has shown that a connective not obeying EC (and EC#) is not necessarily trivial, can have some very interesting properties, and may have some useful applications, in computer sciences, artificial intelligence, etc. (see e.g. [21]).

Before the development of paraconsistent logic, it was possible to claim that a negation not obeying the principle of non contradiction would be absurd, because such a negation would be too weak to deserve the name. This claim was of course not well founded but, at least, there were no results against it. With paraconsistent logic, there are : such a negation is not so weak. On the other hand, it seems that, until now, there are no definitive arguments for the opposite claim according to which operators studied in paraconsistent logic are really negations.

Of course in the philosophical tradition (Heraclitus, Hegel, etc.) and also at a more trivial level (political rethoric, advertisements, fantasms and desires, etc.) there are some uses of the concept of negation which can apparently support the idea of a *paraconsistent* negation. Nevertheless only a precise characterization of what is a negation and a result showing that a paraconsistent negation reflecting these uses is a negation according to this characterization, could be convincing and allow one to righlty speak about a paraconsistent *negation*. [19]

The question of whether a paraconsistent negation is really a negation is not just a question of language. It can be seen as a challenging philosophical question which must be seriously investigated, in particular with the help of technical tools of modern logic.

References

[1] A.Bialynicki-Birula and H.Rasiowa, "On the representation of quasi-boolean algebras", *Bulletin de l'Académie Polonaise des Sciences, Classe III*, **5** (1957), pp.259-261.

[2] A.I.Arruda, 1984, "N.A.Vasiliev : a forerunner of paraconsistent logic", *Philosophia Naturalis*, **21**, 472-491.

[3] J.-Y.Béziau, 1990, "Logiques construites suivant les méthodes de da Costa", *Logique et Analyse*, **131-132**, 259-272.

[4] J.-Y.Béziau, 1993, "Nouveaux résultats et nouveau regard sur la logique paraconsistante C1", *Logique et Analyse*, **141-142**, 45-58.

[5] J.-Y.Béziau, 1994, "Universal logic", in *Logica'94 - Proceedings of the 8th International Symposium*, T.Childers and O.Majer (eds), Czech Academy of Sciences, Prague, pp.73-93.

[19] It seems that the process of formalization is an important difficulty for these kinds of investigations. Paraconsistent logic, as it is understood today, rejects one fundamental property of classical logic, nevertheless it has many common features with classical logic : the distance between modern logic and the Aristotelean theory of syllogism seems, in a sense, bigger that the distance between classical modern logic and paraconsistent logic, the same can be said about the distance relatively to "informal" logic. That's why we can have some doubts about the already existing paraconsistent formalizations of e.g. Hegel's dialectics in exactly the same way that we would have some doubts about the formalization of Kant's philosophy by means of modern classical logic. To be successful in this line of investigations, paraconsistent logic would have to combine not only with other standard non classical logics, such as e.g. relevant logic, but also with "natural" logics which challenge some fundamental features of modern logic funding the usual relations between modern logic and language and/or thought.

[6] J.-Y.Béziau, 1994, "Théorie législative de la négation pure", *Logique et Analyse*, **147-148**, 209-225.

[7] J.-Y.Béziau, 1995, "Negation : what it is and what it is not", *Boletim da Sociedade Paranaense de Matemática*, **15**, 37-43.

[8] J.-Y.Béziau, 1997, "Logic may be simple", *Logic and Logical Philosophy*, **5**, 129-147.

[9] J.-Y.Béziau, 1998, "Idempotent full paraconsistent negations are not algebraizable", *Notre Dame Journal of Formal Logic*, **39**, 135-139.

[10] J.-Y.Béziau, 1999, "Classical negation can be expressed by one of its halves", *Logic Journal of the Interest Group on Pure and Applied Logics*, **7**, 145-151.

[11] J.-Y.Béziau, 1999, "Paraconsistent logic !", to appear.

[12] J.-Y.Béziau, 1999, "S5 is a paraconsistent logic and so is first-order classical logic", to appear.

[13] A.Bobenrieth, 1996, *¿Inconsistencias, Por qué no?*, Colcultura, Bogota.

[14] W.Bychovski, 1987, "Polar paraconsistent logic", *Proceedings of the Poldavian Academy of Sciences*, **37**, 123-156.

[15] N.C.A. da Costa, 1958, "Nota sobre o conceito de contradição", *Anuário da Sociedade Paranaense de Matemática*, **1**, 6-8.

[16] N.C.A. da Costa, 1963, "Calculs propositionels pour les systèmes formels inconsistants", *Comptes Rendus de l'Académie des Sciences de Paris*, **257**, 3790-3793.

[17] N.C.A. da Costa, 1966, "Opérations non monotones dans les treillis", *Comptes Rendus de l'Académie des Sciences de Paris*, **263A**, 429-432.

[18] N.C.A. da Costa, 1997, *Logiques classiques et non classiques*, Masson, Paris.

[19] N.C.A. da Costa and J.-Y.Béziau, 1994, "Théorie de la valuation", *Logique et Analyse*, **146**, 95-117.

[20] N.C.A. da Costa and J.-Y.Béziau, 1997, "Overclassical logic", *Logique et Analyse*, **157**, 31-44.

[21] N.C.A. da Costa and V.S.Subrahmanian, 1989, "Paraconsistent logics as a formalism for reasoning about inconsistent knowledge bases", *Artificial Intelligence in Medecine*, **1**, 167-184.

[22] H.B.Curry, 1952, *Leçons de logique algébrique*, Gauthier-Villars, Paris and E.Nauwelaerts, Louvain.

[23] S.Jaśkwoski, 1948, "Rachunek zdań dla systemów dedukcyjnych sprzecznych", *Studia Societatis Scientiarum Toruniensis*, **1A**, 57-77.

[24] I.Johansson, 1936, "Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus", *Compositio Mathematicae*, **4**, 119-136.

[25] W.Kneale and M.Kneale, 1962, *The development of logic*, Clarendon, Oxford.

[26] S.C.Kleene, 1938, "On a notation for ordinal numbers", *The Journal of Symbolic Logic*, **3**, 150-155.

[27] S.Leśniewski, 1927, "O podstawach matematyki", *Przegląd Filozoficzny*, **30**, 164-206.

[28] L.Lismont, 1992, "Logique du premier ordre anti-fondée", *Cahiers du Centre de Logique*, **7**, 69-95.

[29] J.Łukasiewicz, 1920, "O logice trójwartościowej", *Ruch Filozoficny*, **5**, 170-171.

[30] C.Mortensen, 1989, "Paraconsistency and C1", in [33], pp.289-305.

[31] G.Priest, 1979, "Logic of paradox", *Journal of Philosophical Logic*, **8**, 219-241.

[32] G.Priest and R.Routley, 1989, "Systems of paraconsistent logic", in [33], pp.151-186.

[33] G.Priest, R.Routley and J.Norman (eds), 1989, *Paraconsistent logic : Essays on the inconsistent*, Philosophia, Munich.

[34] L.Puga and N.C.A. da Costa, 1988, "On the imaginary logic of N.A.Vasiliev", *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, **34**, 205-211.

[35] A.P.Pynko, 1995, "On Priest's logic of paradox", *Journal of Applied Non-Classical Logics*, **5**, 219-225.

[36] A.M.Sette, 1973, "On the propositional calculus P1", *Mathematica Japonae*, **16**, 173-180.

[37] B.H.Slater, 1995, "Paraconsistent logics ?", *Journal of Philosophical Logic*, **24**, 451-454.

[38] R.Sylvan, 1990, "Variations on da Costa C-systems and dual-intuitionistic logics. 1. Analyses of C_ω and CC_ω", *Studia Logica*, **49**, 47-65.

[39] R.Sylvan and I.Urbas, 1993, "Paraconsistent classical logic", *Logique et Analyse*, **141-142**, 3-24.

[40] A.Tarski, 1936, "O pojciu wynikania logicznego", *Przegląd Filozoficzny*, **39**, 58-68.

[41] I.Urbas, 1987, *On Brazilian paraconsistent logics*, PhD, Canberra, 1987.

[42] I.Urbas, 1990, "Paraconsistency", *Studies in Soviet Thought*, **39**, 343-354.

[43] I.Urbas, 1996, "Dual-intuitionistic logic", *Notre Dame Journal of Formal Logic*, **37**, 440-451.

[44] M.Urchs, 1995, "Discursive logic. Towards a logic of rational discourse", *Studia Logica*, **54**, 231-249.

[45] R.Wójcicki, 1988, *Theory of logical calculi*, Reidel, Dordrecht.

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